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QED₂ and U(1)-Problem

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Abstract

QED_2 with mass and N flavors of fermions is constructed using Euclidean path integrals. The fermion masses are treated perturbatively and the convergence of the mass perturbation series is proven for a finite space-time cutoff. The expectation functional is decomposed into clustering θ -vacua and their properties are compared to the θ -vacua of QCD for zero fermion mass. The sector that is created by the N^2 classically conserved vector currents is identified. The currents that correspond to a Cartan subalgebra of $U(N)$ are bosonized together with the chiral densities in terms of a generalized Sine-Gordon model. The solution of the $U(1)$ -problem of QED_2 is discussed and a Witten-Veneziano formula is shown to hold for the mass spectrum of the pseudoscalars. Evaluation of the Fredenhagen-Marcu confinement order parameter clarifies the structure of superselection sectors.

Contents

1	Introduction	3
1.1	Prologue	3
1.2	Overview	4
2	The three mysteries	7
2.1	θ -vacuum	7
2.2	U(1)-problem	11
2.3	Witten-Veneziano type formulas	15
3	QED₂	19
3.1	Formal description of the model	19
3.2	Symmetry properties of the model	21
3.3	Topologically nontrivial configurations for U(1) gauge fields in two dimensions	22
3.4	Outline of the construction	24
4	Construction of the massless model	27
4.1	The fermion determinant	27
4.2	Remarks on the massive case	31
4.3	Measures for A and h	33
4.4	The propagator and the fields φ and θ	36
5	Decomposition into clustering states and the vacuum angle	38
5.1	Clustering and the uniqueness of the vacuum	38
5.2	Identification of operators that violate clustering	39
5.3	Symmetry properties of operators that violate clustering	46
5.4	Decomposition into clustering states	47
6	Bosonization and vector currents	52
6.1	Evaluation of a generalized generating functional	52
6.2	Bosonization prescription	59
6.3	More vector currents	64
6.4	Theorems on n-point functions	67

7	The generalized Sine Gordon model	72
7.1	Definition of the model	72
7.2	Convergence of the mass perturbation series	74
7.3	Remarks on the mass perturbation	87
7.4	Semiclassical approximation	94
7.5	Witten-Veneziano formula	100
8	Confinement in the massless model	103
	Summary	112
	Acknowledgements	115
A	The field theory appendix	116
A.1	Propagators in two dimensions	116
A.2	Gaussian measures	118
A.3	Finite action is zero measure	119
A.4	Wick ordering and massless particles	120
B	The technical appendix	124
B.1	Notational conventions	124
B.2	Some integrals	125
B.3	Some matrices	126
B.4	Inverse conditioning	128
B.5	Conditioning	129
B.6	Dirichlet boundary conditions	130
B.7	A generalized Hölder inequality	132
B.8	Bound on integrals over Cauchy determinants	135
	Bibliography	136

Chapter 1

Introduction

1.1 Prologue

It is well known that the mathematical structure of four dimensional (4d), realistic field theories is much more involved than the world of 2d models. Therefore there is a long history of attempts to study physical problems in low dimensional theories. Many concepts have first been developed in 2d toy models before they were taken over to 4d physics.

Maybe the most prominent example is U(1)-gauge theory in two dimensions first analyzed by Schwinger [52] and therefore christened *Schwinger model*. A very attractive feature of the model is its rather simple solution as long as there is no mass term taken into account. The construction of the massive model [20], [27] is a little bit more subtle. Also the introduction of several flavors [10], [28], [41] makes the model less straightforward than the original version. Nevertheless it is rather surprising that the case of more than one flavor has not been analyzed systematically yet. It is the intent of this thesis to fill this gap and to push forward the construction of QED in two dimensions with mass and flavor (QED₂) as far as possible.

Of course this project is inspired by some ‘4d mysteries’, as should be any investigation of toy models. Namely the topics that will be attacked are the construction of the θ -vacuum in QCD, the U(1)-problem and Witten-Veneziano type formulas. Those problems are closely related to each other.

The θ -vacuum [15], [39] is supposed to be the superposition of topological sectors in order to obtain the gauge invariant, physical vacuum. As will be discussed below, the mathematical status of this construction is rather vague. Nevertheless the θ -vacuum is a generally accepted concept. In particular it was used to propose a solution of the U(1)-problem [36], [37].

Due to the breaking of the axial U(1)-symmetry one could a priori expect a corresponding Goldstone boson. The lack of experimental evidence for this particle is referred to as the U(1)-problem [67]. At first glance this problem does not seem to be there at all, since the U(1) axial current acquires the Adler-

Bardeen anomaly when quantizing the theory [2], [6], [9], [53]. Using the fact that the anomaly can be rewritten as a total divergence [7], the current can be redefined in such a way that it is conserved. Ignoring the missing gauge invariance of the newly defined current, one now can indeed expect a Goldstone particle. From a less reckless point of view it has to be doubted if the U(1)-problem is really well posed, since gauge invariance is one of the corner-stones of QCD, and the gauge variant conserved current does not act in the physical subspace.

Finally Witten-Veneziano type formulas [59], [63], [69] are another link between the topologically nontrivial structure of the QCD vacuum and the U(1)-problem. They relate the masses of pseudoscalar mesons (the 'would be Goldstone bosons') to the topological susceptibility. Unfortunately the status of those formulas is not completely clear, or they are only formulated for massless QCD.

The three quoted problems can all be addressed rather well in QED₂. U(1)-gauge theory in two dimensions has a nontrivial topological structure and the formal construction of the θ -vacuum can be performed. The axial-vector current has an anomaly, and the Schwinger model shows mass generation. Thus the situation concerning the U(1)-problem is equivalent to QCD. Finally Witten-Veneziano formulas should be obeyed as well.

Of course it would not make sense to repeat the argumentation from QCD. Here the strategy will be to construct the model independent of poorly defined concepts like θ -vacua, and to draw the lessons for QCD afterwards. On the way also a new and careful construction of θ -vacua will be given.

1.2 Overview

Before I start to explore what has been outlined in the prologue, a short overview will be given.

To be more explicit about what should be learned for QCD, the announced 4d topics will be discussed in Chapter 2. I will review the construction of the θ -vacuum, the U(1)-problem and Witten-Veneziano type formulas. In particular it will be pointed out where criticism is advisable.

The formulation that will be used to construct QED₂ is the framework of Euclidean path integrals. Section 3.1 is dedicated to the discussion of the Euclidean action that describes the model under consideration. Besides the gauge field it will be necessary to introduce another vector field h_μ . This auxiliary field generates a Thirring term (current-current interaction) that is needed for a proper treatment of the mass term. In order to ensure that QED₂ is appropriate for analyzing the above mentioned problems, the symmetry properties of the model will be discussed in 3.2. This is followed by Section 3.3 where topological properties of U(1) gauge fields in two dimensions are

reviewed.

In the usual approach (which is adopted here), one first integrates out the fermions. This gives rise to the fermion determinant which is discussed in Chapter 4. It will turn out that it does not have a simple structure when the fermions are massive. Thus the strategy will be mass perturbation theory. The basic formulas needed for this enterprise are derived in Section 3.4 .

Chapter 4 is dedicated to giving a precise mathematical definition of the so far poorly defined path integral. To this end I first elaborate on the fermion determinant in an external field (Sections 4.1, 4.2). It will turn out that for massless fermions the fermion determinant is Gaussian in the external field (compare e.g. [56]). Together with the action for the gauge field and auxiliary field, respectively, this will amount to common Gaussian measures which have a precise mathematical meaning (Section 4.3). In two dimensions gauge fixing can be used to reduce the gauge field to one scalar degree of freedom. It is more convenient to work with those scalar fields which are introduced in Section 4.4 where I also rewrite the fermion propagator in terms of those variables.

A proper field theory has to obey the cluster decomposition property (5.1) in order to guarantee the existence of a unique vacuum state. It turns out that for the expectation functional so far constructed clustering is violated by a certain class of operators which I classify in 5.2. Using the symmetry properties (Section 5.3) of those operators the state can be decomposed into clustering θ -vacua that are introduced in 5.4 . Furthermore it is proven that the new state defines a proper field theory. This decomposition into clustering states is exactly what is hoped to have been obtained in QCD by introducing the θ -vacuum. Several similarities between the two constructs will be discussed.

In two dimensions one has the elegant technique of bosonization at hand. This means that the vacuum expectation values of certain operators can be expressed by vacuum expectation values in a bosonic theory. In Section 6.1 I evaluate a generalized generating functional in the massless model which then can be mapped onto a theory of free bosons which is described in 6.2. In particular the vector currents that are diagonal in flavor space (*Cartan type currents*) have a simple transcription in the bosonic theory. Anyhow one can define currents for all generators of $U(N)_{\text{flavor}}$, but it is not possible to find a local bosonization for the whole set of vector currents as I will show in 6.4 . There I also discuss the Hilbert space of the states described by the currents. Having established the bosonized version of the model, it is rather easy to analyze the status of the $U(1)$ -problem of QED_2 .

By summing up the mass perturbation series one can construct a theory that bosonizes the states described by the Cartan currents in terms of a generalized Sine Gordon theory. This procedure is a generalization of the Coleman isomorphism [20] and will be discussed in Section 7.1 . In Section 7.2 I prove that the mass perturbation series converges if a space-time cutoff Λ is imposed. Due to the presence of massless degrees of freedom in the bosonic model for

more than one flavor the known methods to remove Λ termwise fail. This rather unpleasant feature of the multiflavor model will be discussed in Section 7.3 . Nevertheless one can extract interesting physical information (vacuum structure, mass spectrum) from a semiclassical approximation of the model (Section 7.4). This semiclassical spectrum will then be used to test Witten-Veneziano formulas (7.5).

In Chapter 8 a generalized version of the Fredenhagen-Marcu parameter [24] will be computed, and the confinement properties of the model will be analyzed.

In a short summary the obtained results will be discussed.

Finally I announce the appendices. Appendix A collects material that can be found in the literature, but is included to keep the thesis self contained. In particular propagators in two dimensions, Gaussian measures and Wick ordering will be discussed. Also a toy example can be found there which illustrates that the set of fields having finite action has measure zero. Appendix B contains formulas that are of more technical nature, and thus were not included in the main part.

Chapter 2

The three mysteries

The aim of this chapter is to prepare the physical playground for the toy model. I review the three 'QCD-topics' whose analogues will be analyzed in QED₂ and point out where criticism is advisable.

2.1 θ -vacuum

Almost twenty years ago it was realized by Belavin et al. [8], that classical Yang-Mills theory in Euclidean space allows for topologically nontrivial solutions called *instantons*¹. They were obtained by analyzing gauge field configurations with finite action. A sufficient condition for such configurations is to approach a pure gauge when one sends the space-time argument to infinity

$$A_\mu(x) \xrightarrow{x^2 \rightarrow \infty} -\frac{i}{g} \left(\partial_\mu S(x) \right) \left(S(x) \right)^{-1}, \quad (2.1)$$

where $S(x)$ are elements of SU(N). The instanton solutions $A^{(n)}$ can be classified with respect to their winding number $\nu[A]$ (Pontryagin index, Chern Number, see e.g. [64]), which takes on integer values

$$\nu[A^{(n)}] = n, \quad (2.2)$$

where

$$\nu[A] := \frac{g^2}{16\pi^2} \int d^4x \text{Tr} \left(F_{\mu\nu} \tilde{F}_{\mu\nu} \right), \quad (2.3)$$

and

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]. \quad (2.4)$$

¹ The discussion below can be repeated for U(1) gauge fields in two dimensions, where the topological objects are considerably easier to imagine. I already announce here that this 2d material will be presented in Section 3.3, where I will be more explicit on the topologically nontrivial configurations and the topological index. For the reader not so much familiar with instantons, this will be a nice illustration of the material presented here in a rather compressed form.

There is an important identity [7], [17] (Bardeen's identity)

$$\frac{1}{2} \text{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) = \partial_\mu K_\mu, \quad (2.5)$$

where

$$K_\mu = \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left(A_\nu \partial_\rho A_\sigma - i \frac{2g}{3} A_\nu A_\rho A_\sigma \right). \quad (2.6)$$

Together with the Gauss theorem, (2.5) can be used to rewrite the winding number as a surface integral

$$\nu[A] = \lim_{V_4 \rightarrow \infty} \frac{g^2}{8\pi^2} \int_{V_4} d^4x \partial_\mu K_\mu = \lim_{V_4 \rightarrow \infty} \frac{g^2}{8\pi^2} \int_{\partial V_4} d^3\sigma \hat{n}_\mu K_\mu. \quad (2.7)$$

The topologically nontrivial configurations radically change the nature of the vacuum. The standard argument [15], [39] is formulated in temporal ($A_4 = 0$) gauge. The instanton $A^{(n)}$ in temporal gauge has the property

$$A_i^{(n)}(x) \longrightarrow \begin{cases} -\frac{i}{g} \left(\partial_i S^{(m)}(\vec{x}) \right) \left(S^{(m)}(\vec{x}) \right)^{-1} & \text{for } x_4 \rightarrow -\infty \\ -\frac{i}{g} \left(\partial_i S^{(m+n)}(\vec{x}) \right) \left(S^{(m+n)}(\vec{x}) \right)^{-1} & \text{for } x_4 \rightarrow +\infty, \end{cases} \quad (2.8)$$

$i = 1, 2, 3$ and $n, m \in \mathbb{Z}$.

$$S^{(1)}(\vec{x}) = \frac{\vec{x}^2 - \lambda^2}{\vec{x}^2 + \lambda^2} - \frac{2i\lambda\vec{\sigma} \cdot \vec{x}}{\vec{x}^2 + \lambda^2}, \quad (2.9)$$

where σ_i , $i = 1, 2, 3$ are the Pauli matrices and λ is a real number, the *instanton size*. For $S^{(l)}(x)$ with $l \neq 1$ see e.g. [48]. Due to the ε -tensor in Equation (2.6) only K_4 can contribute in the surface integral (2.7). Choosing the surface ∂V_4 to be integrated over, to be the hypercylinder of Figure 2.1 (next page), the winding number (2.7) takes on the form

$$\nu[A] = \lim_{T \rightarrow \infty} \lim_{V_3 \rightarrow \infty} \frac{g^2}{8\pi^2} \int_{V_3} d^3x \left(K_4 \Big|_{x_4=T} - K_4 \Big|_{x_4=-T} \right) =: \nu_+[A] - \nu_-[A]. \quad (2.10)$$

From Equation (2.9) equation one easily reads off

$$S^{(1)}(\vec{x}) \xrightarrow{|\vec{x}| \rightarrow \infty} \mathbb{I}, \quad (2.11)$$

and the same is true for all $S^{(l)}$. Thus 3-space can be compactified to the hypersphere S^3 . Since the manifold of $\text{SU}(2)$ is homeomorphic to S^3 , the $S^{(l)}(\vec{x})$ define mappings $S^3 \rightarrow S^3$. Such mappings are known to fall into homotopy classes [64].

Figure 2.1 : The surface ∂V_4 .

By inserting the asymptotic form (2.8) for $A^{(n)}$ into (2.6) one obtains (similar for $\nu_+[A^{(n)}]$)

$$\begin{aligned} \nu_-[A^{(n)}] = \\ \frac{1}{24\pi^2} \int_{S^3} d^3x \, \varepsilon_{ijk} \text{Tr} \left[(\partial_i S^{(m)}) (S^{(m)})^{-1} (\partial_j S^{(m)}) (S^{(m)})^{-1} (\partial_k S^{(m)}) (S^{(m)})^{-1} \right] = m . \end{aligned} \quad (2.12)$$

The left hand side of (2.12) is known to be the integral over the invariant measure of the group (see e.g. [48]), and thus $\nu_-[A^{(n)}]$ and $\nu_+[A^{(n)}]$ give the homotopy classes of the gauge field configurations $A^{(n)}$ at time equal to plus and minus infinity. Looking at (2.10) and (2.12) one now can interpret the instanton $A^{(n)}$ obeying (2.8) in the following way².

The instanton $A^{(n)}$ with total winding number $\nu[A^{(n)}] = n$ connects a pure gauge at time equal to minus infinity that winds $\nu_-[A^{(n)}] = m$ times around compactified space, with a pure gauge at time equal to plus infinity that winds $\nu_+[A^{(n)}] = m + n$ times around compactified space.

So far the instanton has only been constructed for the gauge group SU(2). Of course one is interested in SU(3) when dealing with QCD. In [21] it is

²I did not denote the explicit form for $A^{(n)}$ (it can be found in e.g. [48]) since it is rather lengthy. Only the asymptotic form (2.8) is quoted. For the 2d case the explicit form will be given in 3.3.

discussed that mappings from S^3 to $SU(3)$ can be deformed continuously into a mapping from S^3 to a $SU(2)$ subgroup. Thus the $SU(2)$ discussion is already sufficient for $SU(3)$.

The configurations

$$-\frac{i}{g} \left(\partial_i S^{(l)}(\vec{x}) \right) \left(S^{(l)}(\vec{x}) \right)^{-1}, \quad (2.13)$$

for infinite time argument are now considered as classical vacuum states $|l\rangle$. The reasoning therefore is the following. The classical vacua have zero potential energy separated by a barrier [39]. On the other hand the instanton $A^{(1)}$ that connects $|m\rangle$ with $|m+1\rangle$ has Euclidean action $8\pi^2/g^2$. Thus in the WKB sense the $|m\rangle$ to $|m+1\rangle$ amplitude is of order $\exp(-8\pi^2/g^2)$ which is a typical tunneling amplitude [15].

A general transition from $|m\rangle$ to $|m+n\rangle$ can formally be expressed in terms of functional integrals

$${}_{+\infty}\langle m+n|m\rangle_{-\infty}^J = \frac{1}{Z} \int \mathcal{D}A \delta(\nu[A] - n) \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_J}. \quad (2.14)$$

The left hand side denotes the transition from the vacuum $|m\rangle$ at (Euclidean) time equal to minus infinity to the vacuum $|m+n\rangle$ at time equal to plus infinity in the presence of sources J . Since the instanton $A^{(n)}$ was identified to mediate such a transition (see (2.10)), one has to integrate over gauge field configurations within the instanton sector with winding number n . This is formally expressed by the δ -functional in the path integral. Finally S_J denotes the Euclidean action plus coupling terms to the sources J . Obviously the expression (2.14) does not depend on the vacuum $|m\rangle$ I started with, but only on the difference n .

Since transitions between the vacua $|l\rangle$ are possible, none of them can be the correct vacuum. The crucial idea in the construction of the θ -vacuum $|\theta\rangle$ is to form a superposition that takes into account all possible transitions. In terms of functional integrals this can be expressed as (using (2.3) in the last step)

$$\begin{aligned} {}_{+\infty}\langle \theta|\theta\rangle_{-\infty}^J &:= \sum_{n=-\infty}^{n=+\infty} e^{i\theta n} {}_{+\infty}\langle m+n|m\rangle_{-\infty}^J \\ &= \frac{1}{Z} \int \mathcal{D}A \sum_{n=-\infty}^{n=+\infty} \delta(\nu[A] - n) e^{i\theta\nu[A]} \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_J} \\ &= \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left(i\theta \frac{g^2}{16\pi^2} \int d^4x \text{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) - S_J \right). \end{aligned} \quad (2.15)$$

Formally (see e.g. [45]) the θ -vacuum can be written as a vector in a Hilbert space

$$|\theta\rangle = \sum_{l=-\infty}^{l=+\infty} e^{-i\theta l} |l\rangle. \quad (2.16)$$

Thus θ is only defined mod(2π). The gauge transformation $S^{(1)}$ (see (2.9)) that changes $|l\rangle$ to $|l+1\rangle$ can formally be implemented as an unitary operator $\hat{S}^{(1)}$ in this Hilbert space [15], and now leaves $|\theta\rangle$ invariant up to a phase

$$\hat{S}^{(1)} |\theta\rangle = e^{i\theta} |\theta\rangle. \quad (2.17)$$

As can be seen from (2.15), a new term has entered the action, namely the winding number times a new parameter, the *vacuum angle* θ . This term causes a serious difficulty, since it violates CP invariance. This problem is known as the *strong CP problem* (see e.g. [45]) for a nice review). There is a second point in the construction of $|\theta\rangle$ which is rather problematic. It has been pointed out (compare the second line of (2.15)), that the measure has to be a sum over all topological sectors. This requirement is not really mathematically well defined. Functional measures for gauge fields (if they are constructed at all) do not live on continuous functions and configurations with finite action have measure zero. The first point can be seen in Appendix A.2 on Gaussian measures, which are measures on the space of tempered distributions. Gaussian measures are in fact a good illustration, since 2d, U(1) gauge theory makes use of them. The second objection is illustrated in Appendix A.3 where I show for a toy example that configurations with finite action have zero measure.

Nevertheless the θ -vacuum is a widely accepted concept, and was e.g. invoked to solve the U(1)-problem. This will be discussed in the next two sections.

I finish this section with remarking that it is generally believed that physics does not depend on θ , whenever one of the quarks is massless. The arguments for this make use of the anomaly and will be discussed in the next section.

2.2 U(1)-problem

The symmetry of the QCD Lagrangian with three flavors and vanishing quark masses is $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$. In the following I review the argumentation that this symmetry is spontaneously broken down to $SU(3)_V \times U(1)_V$ and Goldstone particles have to be expected. In the discussion below, I use the notation of [12].

Consider the one parameter axial transformations

$$q \rightarrow g^{(a)} q \quad , \quad \bar{q} \rightarrow \bar{q} g^{(a)} \quad , \quad (2.18)$$

where q is the triplet $(u, d, s)^T$ and

$$g^{(a)} = e^{i\omega^{(a)}\tau^{(a)}\gamma_5} \quad , \quad a \text{ fixed} \quad ,$$

$$\tau^{(1)} = \mathbb{I} \quad , \quad \tau^{(a)} = \frac{\lambda^{(a-1)}}{2} \quad , \quad a = 2, 3, \dots, 9 \quad , \quad (2.19)$$

where $\lambda^{(b)}$, $b = 1, 2, \dots, 8$ are the Gell-Mann matrices. As long as the quark masses vanish, this is a symmetry of the QCD Lagrangian. The corresponding Hermitean Noether currents are

$$j^{(1)\mu}_5 := \bar{q} \gamma^\mu \gamma_5 q ,$$

$$j^{(a)\mu}_5 := \bar{q} \gamma^\mu \gamma_5 \frac{\lambda^{(a-1)}}{2} q , \quad a = 2, 3, \dots, 9 . \quad (2.20)$$

The U(1) axial current $j^{(1)}_5$ acquires the anomaly when quantizing the theory and thus will be discussed later. The other currents are conserved

$$\partial_\mu j^{(a)\mu}_5(x) = 0 , \quad a = 2, 3, \dots, 9 . \quad (2.21)$$

To each conserved current one can define an operator $D^{(a)}$ acting on a polynomial X of the fields via

$$D^{(a)}(X) := i \int_{x^0=\text{const}} [j^{(a)0}_5(x) , X] d^3x . \quad (2.22)$$

The integrals converge due to local commutativity, and are time independent due to conservation of the currents [12]. They immediately can be shown to obey $D^{(a)}(X^*) = D^{(a)}(X)^*$, $D^{(a)}(XY) = D^{(a)}(X) Y + X D^{(a)}(Y)$, and thus are called \star -derivatives. In [12] it is discussed how the \star -derivatives generate internal Lie-group symmetries of the theory³. Now the question is if these symmetries can be implemented unitarily or are spontaneously broken.

First I assume that the unitary implementation is possible. This means that there exists an unitary operator $V(g^{(a)})$ depending continuously on the element $g^{(a)}$ of the symmetry group and the symmetry operates via

$$X \rightarrow V(g^{(a)}) X V(g^{(a)})^{-1} . \quad (2.23)$$

As is discussed in [12] this then implies

$$V(g^{(a)}) = e^{i\omega^{(a)} Q^{(a)}} , \quad (2.24)$$

and

$$Q^{(a)} X |0\rangle = -i D^{(a)}(X) |0\rangle . \quad (2.25)$$

Since the right hand side of (2.22) does not depend on time,

$$[H, Q^{(a)}] = 0 . \quad (2.26)$$

Consider now an eigenstate $|E_n\rangle = X_n |0\rangle$ of the Hamiltonian with energy E_n . Due to (2.26) $Q^{(a)} |E_n\rangle$ is an eigenstate with the same energy

$$H Q^{(a)} |E_n\rangle = E_n Q^{(a)} |E_n\rangle . \quad (2.27)$$

³ If one assumes that the expression $Q := \int j^0(x) d^3x$ exists for a conserved current j^μ , then the unitary operator that performs the underlying symmetry transformation is given by $V_\omega = \exp(i\omega Q)$, and $D(X)$ is the second term in an expansion of $V_\omega X V_\omega^{-1}$ in ω .

Since the axial vector currents have odd parity, a parity transformation \mathcal{P} acts on $Q^{(a)} |E_n\rangle$ via

$$\mathcal{P} Q^{(a)} |E_n\rangle = \mathcal{P} (-i) D^{(a)}(X_n) |0\rangle = -P_n Q^{(a)} |E_n\rangle, \quad (2.28)$$

where P_n denotes the parity of the eigenstate $|E_n\rangle$. Thus if one of the symmetries generated by the conserved currents $j^{(a)}_5$, $a = 2, 3, 9$ were realized unitarily, this would imply that the hadrons come in parity doublets. Since the parity partners are not seen in experiment, the assumption of unitary implementability is wrong, and all those symmetries have to be broken spontaneously.

The breaking of the symmetry for each single parameter group generated by one of the conserved currents $j^{(a)}_5$, $a = 2, 3, 9$ implies the existence of 8 Goldstone particles. The corresponding massless states $|\Phi^{(a)}\rangle$ are connected to the vacuum via

$$\langle \Phi^{(a)} | j^{(a)}_5 | 0 \rangle \neq 0. \quad (2.29)$$

The whole discussion above made use of vanishing quark masses. Since these masses are known to be nonzero, the Goldstone particles are only approximate Goldstone bosons. Nevertheless the pseudoscalar mesons

$$\pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta \quad (2.30)$$

can be properly identified to play this role.

So far the U(1)-current was excluded since it acquires the anomaly [2], [6], [9]

$$\partial_\mu j^{(1)\mu}_5 = 2N_f \frac{g^2}{16\pi^2} \text{Tr}(F^{\mu\nu} \tilde{F}_{\mu\nu}), \quad (2.31)$$

where N_f is the number of flavors. This implies that the action of the corresponding \star -derivative on some polynomial X of the fields $D^{(1)}(X)$ is not time independent.

Using Bardeen's identity ([7], [17], compare (2.5), (2.6))

$$\frac{1}{2} \text{Tr}(F^{\mu\nu} \tilde{F}_{\mu\nu}) = \partial_\mu K^\mu,$$

$$K_\mu = \varepsilon_{\mu\nu\rho\sigma} \text{Tr}\left(A^\nu \partial^\rho A^\sigma - i \frac{2g}{3} A^\nu A^\rho A^\sigma\right) \quad (2.32)$$

one can define a new current

$$\tilde{j}^{(1)\mu}_5 := j^{(1)\mu}_5 - 2N_f \frac{g^2}{8\pi^2} K^\mu, \quad (2.33)$$

which now is conserved

$$\partial_\mu \tilde{j}^{(1)\mu}_5 = 0. \quad (2.34)$$

One can also define a \star -derivative $\tilde{D}^{(1)}$

$$\tilde{D}^{(1)}(X) := i \int_{x^0=const} [\tilde{j}^{(1)0}_5(x), X] d^3x, \quad (2.35)$$

corresponding to the new current $\tilde{j}^{(1)}_5$. All the arguments (2.23) - (2.29) can be repeated and a ninth pseudoscalar particle can be expected. One would like to interpret the η' in that sense. The common wisdom is that η' is too heavy to be this approximate Goldstone boson. This belief is based on a work by Weinberg [67] where the case of two flavors is considered. There the η should play the role of the U(1) Goldstone particle. To interpret the η in this sense, the mass relation $m_\eta < \sqrt{3}m_\pi$ has to be obeyed. Inserting the experimental values for the masses ($m_{\pi^0} \sim 135$ MeV, $m_\eta \sim 549$ MeV), one finds that the η is not the wanted Goldstone particle. The same reasoning can be done for three flavors, and the U(1)-problem can be formulated:

Where is the ninth, light, pseudoscalar meson ?

If one reanalyzes the arguments for the U(1)-current more carefully, one finds that K^μ defined in (2.32) is not gauge invariant, and thus $\tilde{j}^{(1)}_5$ is not a physical operator. As discussed in [42] it is not obvious that local commutativity which is needed to establish the convergence of the integral (2.35) should and can be required for non-physical operators. So it has to be doubted that $\tilde{D}^{(1)}$ is well defined and generates a symmetry on the physical Hilbert space. It is unclear if the U(1)-problem is well posed.

Ignoring this criticism, one can formally define a charge \tilde{Q}_5 that corresponds to the current $\tilde{j}^{(1)}_5$ (compare footnote 3 on page 12)

$$\tilde{Q}_5 := \int_{x^0=const} d^3x \tilde{j}^{(1)0}_5(x). \quad (2.36)$$

This can now be used to argue that physics does not depend on θ if the quarks are massless, as has been mentioned in the last section⁴. As has already been pointed out, \tilde{Q}_5 is not gauge invariant. In particular under the gauge transformation $S^{(1)}$ (see Equation (2.9)) which changes $|m\rangle \rightarrow |m+1\rangle$

$$\begin{aligned} \Delta \tilde{Q}_5 &= -\frac{2N_f}{12\pi^2} \int d^3x \varepsilon_{ijk} \text{Tr} \left[(\partial_i S^{(1)}) (S^{(1)})^{-1} (\partial_j S^{(1)}) (S^{(1)})^{-1} (\partial_k S^{(1)}) (S^{(1)})^{-1} \right] \\ &= -2N_f. \end{aligned} \quad (2.37)$$

In the last equation I used (2.33) and (2.12). Now one can perform a chiral rotation on $|\theta\rangle$

$$|\theta\rangle_\delta := e^{i\delta \tilde{Q}_5} |\theta\rangle_\delta, \quad (2.38)$$

⁴ There is another way to establish this result. θ can also be introduced by modifying one of the mass terms to $\bar{\psi} \exp(i\theta\gamma_5) \psi$ (see [5]). If one of quark masses is zero, this modification vanishes and so the θ dependence.

and use (2.37) to obtain

$$\begin{aligned}\hat{S}^{(1)} |\theta\rangle_\delta &= \hat{S}^{(1)} e^{i\delta\tilde{Q}_5} (\hat{S}^{(1)})^{-1} \hat{S}^{(1)} |\theta\rangle = \\ e^{i\delta\tilde{Q}_5 - i\delta 2N_f} e^{i\theta} |\theta\rangle &= e^{i(\theta - \delta 2N_f)} |\theta\rangle_\delta.\end{aligned}\quad (2.39)$$

Since \tilde{Q}_5 stems from a current which is conserved if the quarks are massless, it is formally time independent, and hence commutes with the Hamiltonian. Thus the chirally rotated state $|\theta\rangle_\delta$ can also serve as 'the vacuum'. From Equation (2.39) it follows that

$$|\theta\rangle_\delta = |\theta - \delta 2N_f\rangle, \quad (2.40)$$

and thus the theories are equivalent for all values of θ . The argument fails for massive quarks, since then $\tilde{j}_5^{(1)}$ is not conserved.

Arguments at the same level of rigor were used by 't Hooft to solve the U(1)-problem [36], [37]. The idea is that the nontrivial structure of the QCD vacuum leads to a vanishing residue of the U(1)-Goldstone pole in propagators of physical (i.e. gauge invariant) operators. The crucial formula (see [45]) for the cancellation of the residue is the following structure for the vacuum expectation value of an operator X in the θ -vacuum

$$\langle\theta|X|\theta\rangle = C_X \exp\left(i\frac{\chi_X}{2N_f}\theta\right), \quad (2.41)$$

with the remarkable property

$$C_X = 0 \quad \text{unless} \quad \chi_X = 2N_f n, \quad n \in \mathbb{Z}, \quad (2.42)$$

where χ_X is the chiral U(1)-charge of X . The last equation can be seen to hold by the following formal arguments [15]. From (2.37) there follows for the operators $\hat{S}^{(m)} = (\hat{S}^{(1)})^m$ and \tilde{Q}_5 that $\hat{S}^{(m)}\tilde{Q}_5(\hat{S}^{(m)})^{-1} = \tilde{Q}_5 - 2N_fm$. If the states for different vacuum topology $|m\rangle$ are defined as $|m\rangle = \hat{S}^{(m)}|0\rangle$ with $\tilde{Q}_5|0\rangle = 0$ then there follows $\tilde{Q}_5|m\rangle = 2N_fm|m\rangle$. However since \tilde{Q}_5 is formally conserved one concludes ${}_{+\infty}\langle m+n|m\rangle_{-\infty} \propto \delta_{n,0}$. In general one must find

$${}_{+\infty}\langle m+n|X|m\rangle_{-\infty} \propto \delta_{n,\nu}, \quad (2.43)$$

where X is an operator of chirality $2N_f\nu$, $\nu \in \mathbb{Z}$. Inserting this into the first line of (2.15) one ends up with (2.41).

2.3 Witten-Veneziano type formulas

In 1979 Witten proposed a formula that relates the mass of the η' meson to the topological susceptibility of quarkless QCD. The remarkable feature of the

result is that it does not make use of instantons. The main ingredient of the proof is that physics does not depend on the vacuum angle θ when massless quarks are present (compare Section 2.2). This observation can be related to the topological susceptibility. Consider the free energy density

$$F := \lim_{V \rightarrow \infty} \frac{\ln(Z_V)}{V} , \quad (2.44)$$

where the partition function in finite volume Z_V is formally defined as

$$Z_V := \int \mathcal{D}A \mathcal{D}\bar{q} \mathcal{D}q \exp \left(i S_V + i \frac{\theta g^2}{16\pi^2} \int_V d^4x \text{Tr}(F \tilde{F}) \right) . \quad (2.45)$$

The second derivative of the free energy with respect to θ gives the topological susceptibility χ_{top}

$$- \left. \frac{d^2 F}{d\theta^2} \right|_{\theta=0} = \chi_{top} , \quad (2.46)$$

with

$$\chi_{top} := \left(\frac{g^2}{16\pi^2} \right)^2 \int d^4x \langle T q(x) q(0) \rangle , \quad q(x) := \text{Tr}(F(x) \tilde{F}(x)) , \quad (2.47)$$

where T denotes time ordering. As discussed, physics does not depend on θ if the quarks are massless. It follows that χ_{top} has to vanish then. More generally Witten considers the propagator $U(k)$

$$U(k) := \int d^4x e^{ikx} \langle T q(x) q(0) \rangle . \quad (2.48)$$

Adopting some $1/N_c$ -expansion arguments [35], [68] the propagator is rewritten as

$$U(k) = \sum_{\text{glueballs}} \frac{N_c^2 a_n^2}{k^2 - M_n^2} + \sum_{\text{mesons}} \frac{N_c c_n^2}{k^2 - m_n^2} . \quad (2.49)$$

The first sum also contributes in a theory without any quarks (massless or massive) and is denoted as $U_0(k)$ then. To lowest order in $1/N_c$, this term does not change if the quarks are coupled. On the other hand the right hand side of (2.49) has to vanish in the presence of massless quarks at $k = 0$. Ignoring the fact that both sums have the same sign, Witten claims that they cancel each other. The condition in lowest order of $1/N_c$ is (which would be mathematically correct if there was an extra minus sign)

$$\frac{N_c c_{\eta'}^2}{m_{\eta'}^2} = U_0(0) . \quad (2.50)$$

Using the anomaly equation (2.31) to rewrite $N_c c_{\eta'}^2$ in terms of the decay constant $f_{\eta'}$, Witten ends up with

$$m_{\eta'}^2 = \frac{4N_f}{f_{\eta'}^2} \chi_{top}^0, \quad (2.51)$$

where N_f is the number of flavors and $\chi_{top}^0 = \left(\frac{g^2}{16\pi^2}\right)^2 U_0(0)$ is the topological susceptibility in pure SU(3) gauge theory. It has to be remarked that $f_{\eta'}$ should be evaluated in QCD with vanishing quark masses as can be seen from the derivation above. From PCAC arguments it follows that $f_{\eta'}$ varies only very slowly in the mass variable (see e.g. [65]) and thus the experimental value can be inserted.

Although the derivation is problematic (besides the sign problem, questions concerning regularization were ignored) the formula seems to have some truth in it. It has been reanalyzed in Euclidean space by Seiler and Stamatescu [59]. They pointed out that $\text{Tr}(F\tilde{F})$ is a composite operator and requires some subtraction procedure, leading to a spectral representation

$$U(k) = P(k^2) - \int_0^\infty \frac{d\rho(\mu^2)}{k^2 + \mu^2}. \quad (2.52)$$

$P(k^2)$ denotes some polynomial in the momentum. This formula has to be compared to (2.49) in the Witten derivation. Two main differences appear. There shows up the contact term $P(k^2)$ which is necessary due to the subtractions. Furthermore there is the negative sign in front of the spectral integral, which is required by reflection positivity and the fact that q is odd under time reflections. Now the right hand side really can vanish for $k \rightarrow 0$ when the quarks are massless. Assuming that $d\rho(\mu^2)$ is dominated by the η' contribution

$$d\rho(\mu^2) = c_{\eta'}^2 \delta(\mu^2 - m_{\eta'}^2) d\mu^2, \quad (2.53)$$

one obtains

$$\frac{c_{\eta'}^2}{m_{\eta'}^2} = P^0(0). \quad (2.54)$$

This is now the correct expression that replaces (2.50). In Witten's result there occurs an extra factor N_c which is only an artefact of his derivation within the $1/N_c$ framework. It vanishes when rewriting the result in terms of physical quantities like decay constants. The final result given by Seiler and Stamatescu reads

$$m_{\eta'}^2 = \frac{4N_f}{f_{\eta'}^2} P^0(0). \quad (2.55)$$

The topological susceptibility in the Witten result (2.51) has been replaced by the contact term $P^0(0)$ of the two point function of the topological charge

in the theory with vanishing quark masses⁵. There are two more articles on Witten-Veneziano type formulas I would like to mention. The Witten result was rederived (agreeing on the right hand side) by Veneziano [66]. The approach adopted there is the analysis of anomalous Ward identities in the $1/N_c$ expansion.

The second paper by Smit and Vink [63] is an investigation of the problem in an Euclidean lattice formulation. In this approach the regularization procedure is rather straightforward. The problem is the adequate matching of the lattice quantities to their continuum counterparts. The final result for three flavors given by Smit and Vink reads

$$m_{\eta'}^2 - \frac{1}{2}m_\eta^2 - \frac{1}{2}m_{\pi^0}^2 = \frac{12}{f_\pi^2} \bar{\chi}, \quad (2.56)$$

where

$$\bar{\chi} := \lim_{V \rightarrow \infty} \frac{\kappa_P^2 m_a m_b}{V} \langle \text{Tr}(\gamma_5 G_{aa}) \text{Tr}(\gamma_5 G_{bb}) \rangle_U^{pbc}. \quad (2.57)$$

Here G_{aa} is the fermion propagator for fixed flavor a in an external field $G_{aa}(x, y) = \langle \psi_a(x) \bar{\psi}_a(y) \rangle_\psi$. The trace is over all indices except flavor. κ_P is a renormalization factor that approaches 1 in the continuum limit. Finally $\langle .. \rangle_U^{pbc}$ denotes the expectation value with respect to the gauge fields in a quenched approximation. Periodic boundary conditions are imposed for gauge invariant quantities. $\bar{\chi}$ is formally related to the topological susceptibility through the index theorem [4] (see also [58]). In the continuum

$$\text{Tr}(\gamma_5 G_{aa}) = \frac{Q}{m_a}, \quad (2.58)$$

where Q is the topological charge (now it is also obvious that $\bar{\chi}$ does not depend on the choice of a and b).

⁵ It will turn out that at least in the Schwinger model $P^0(0)$ can be interpreted as the quenched topological susceptibility, so that Witten's formula is recovered.

Chapter 3

QED₂

In this chapter the Lagrangian of the model and its symmetries will be discussed. Furthermore I elaborate on the existence of topologically nontrivial configurations, thus showing that QED₂ is adequate for the study of the problems announced in Chapter 2. Finally the strategy for the construction of the model will be outlined.

3.1 Formal description of the model

The Euclidean action of the model that will be constructed is given by

$$S[\bar{\psi}, \psi, A, h] = S_G[A] + S_h[h] + S_F[\bar{\psi}, \psi, A, h] + S_M[\bar{\psi}, \psi] . \quad (3.1)$$

The gauge field action reads

$$S_G[A] = \int d^2x \left(\frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} \lambda \left(\partial_\mu A_\mu(x) \right)^2 \right) . \quad (3.2)$$

A gauge fixing term is included that will be considered in the limit $\lambda \rightarrow \infty$ which ensures $\partial_\mu A_\mu(x) = 0$ (*transverse* or *Landau gauge*). As usual $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$, denotes the field strength tensor.

In addition to the gauge field an auxiliary field h_μ with action

$$S_h[h] = \frac{1}{2} \int d^2x h_\mu(x) \left(\delta_{\mu\nu} - \lambda' \partial_\mu \partial_\nu \right) h_\nu(x) , \quad (3.3)$$

has to be introduced which generates the announced Thirring term. $S_h[h]$ is simply $\delta_{\mu\nu}$ plus a term that makes h_μ transverse in the limit $\lambda' \rightarrow \infty$. I postpone the discussion of the role of h_μ until the fermion action has been introduced.

The fermion action is a sum over N flavor degrees of freedom

$$S_F[\bar{\psi}, \psi, A, h] = \sum_{b=1}^N \int d^2x \bar{\psi}^{(b)}(x) \gamma_\mu \left(\partial_\mu - ie A_\mu(x) - i\sqrt{g} h_\mu(x) \right) \psi^{(b)}(x) . \quad (3.4)$$

The auxiliary field h_μ is coupled in the same way as the gauge field. $\bar{\psi}^{(b)}$ and $\psi^{(b)}$ are independent Grassmann variables.

Since the mass term will be treated differently from the rest of the fermion action I denote it separately

$$S_M[\bar{\psi}, \psi] = - \sum_{b=1}^N m^{(b)} \int_{\Lambda} d^2x t(x) \bar{\psi}^{(b)}(x) \psi^{(b)}(x) . \quad (3.5)$$

$m^{(b)}$ are the fermion masses for the various flavors. For technical reasons the mass term has to be smeared with a test function t with compact support Λ . In some of the results it will be possible to set t equal to one.

I would like to remark that the model with zero fermion mass (all $m^{(b)} = 0$), is of interest on its own. It will be referred to as the *massless model*. Features of this model will be discussed during the approach to the massive case and in particular in Chapter 8.

Now one can discuss the role of h_μ . The fermion action couples h_μ to the vector currents

$$j_\mu^{(b)}(x) := \bar{\psi}^{(b)}(x) \gamma_\mu \psi^{(b)}(x) \quad (3.6)$$

via

$$- i\sqrt{g} \sum_{b=1}^N \int d^2x h_\mu(x) j_\mu^{(b)}(x) . \quad (3.7)$$

Since $S_h[h]$ gives rise to a Gaussian measure one can integrate out the auxiliary field and obtain a new term $S_T[\bar{\psi}, \psi]$ contributing to the fermion action which replaces $S_h[h]$. It is given by

$$S_T[\bar{\psi}, \psi] = \frac{1}{2} g \sum_{b,b'=1}^N \int d^2x j_\mu^{(b)}(x) (\delta_{\mu\nu} - \lambda' \partial_\mu \partial_\nu)^{-1} j^{(b')}_\nu(x) . \quad (3.8)$$

It is easy to check that the covariance operator corresponding to $S_h[h]$ is given by

$$(\delta_{\mu\nu} - \lambda' \partial_\mu \partial_\nu)^{-1} = \delta_{\mu\nu} + (1 - \lambda' \Delta)^{-1} \lambda' \partial_\mu \partial_\nu . \quad (3.9)$$

In the limit $\lambda' \rightarrow \infty$ this reduces to

$$\lim_{\lambda' \rightarrow \infty} (\delta_{\mu\nu} - \lambda' \partial_\mu \partial_\nu)^{-1} = \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Delta} =: T_{\mu\nu} . \quad (3.10)$$

$T_{\mu\nu}$ obeys the projector relation

$$T^2 = T , \quad (3.11)$$

and projects on the transverse direction, as was expected, since the transverse gauge is being used. Defining now the *$U(1)$ -current*

$$J_\mu^{(1)}(x) := \frac{1}{\sqrt{N}} \sum_{b=1}^N j_\mu^{(b)}(x) , \quad (3.12)$$

the new term of the fermion action can be written as

$$S_T[\bar{\psi}, \psi] = \frac{gN}{2} \int d^2x J_\mu^{(1)}(x)^T J_\mu^{(1)}(x)^T, \quad (3.13)$$

where $J_\mu^{(1)T}$ denotes the transverse projection of $J_\mu^{(1)}$

$$J_\mu^{(1)}(x)^T := T_{\mu\nu} J_\nu^{(1)}(x). \quad (3.14)$$

Obviously h_μ generates a Thirring term for the transverse part of the U(N) flavor singlet current $J_\mu^{(1)}$.

The purpose of this Thirring term is to make the short distance singularity of

$$\bar{\psi}^{(b)}(x) \psi^{(b)}(x) \bar{\psi}^{(b)}(y) \psi^{(b)}(y), \quad (3.15)$$

integrable. The quoted expression is a typical term showing up in a power series expansion of the mass term (3.5). It has to be integrated over $d^2x d^2y$ which is possible only if an ultraviolet regulator such as the Thirring term is included.

3.2 Symmetry properties of the model

This section is devoted to the discussion of the symmetry properties of the model. To make my notations clear, the symmetry generators act as follows

$$\begin{aligned} \text{SU(N)}_L : \quad \psi &\rightarrow \exp\left(i \sum_{\alpha} \omega^{(\alpha)} \tau^{(\alpha)} P_L\right) \psi, \\ \text{SU(N)}_R : \quad \psi &\rightarrow \exp\left(i \sum_{\alpha} \omega^{(\alpha)} \tau^{(\alpha)} P_R\right) \psi, \\ \text{U(1)}_V : \quad \psi &\rightarrow \exp(i\omega \mathbb{I}) \psi, \\ \text{U(1)}_A : \quad \psi &\rightarrow \exp(i\omega \mathbb{I} \gamma_5) \psi. \end{aligned} \quad (3.16)$$

The $T^{(\alpha)}$ denote the SU(N) generators, ω and $\omega^{(\alpha)}$ are real coefficients, and

$$P_L := \frac{1}{2}(1 - \gamma_5), \quad P_R := \frac{1}{2}(1 + \gamma_5). \quad (3.17)$$

For vanishing fermion masses $m^{(b)}$, the Lagrangian of the model has the symmetry $\text{SU(N)}_L \times \text{SU(N)}_R \times \text{U(1)}_V \times \text{U(1)}_A$ as is the case for QCD. When quantizing the massless theory the axial U(1)-current

$$j_{5\mu}(x) := \sum_{b=1}^N \bar{\psi}^{(b)}(x) \gamma_\mu \gamma_5 \psi^{(b)}(x), \quad (3.18)$$

acquires the anomaly

$$\partial_\mu j_{5\mu}(x) = 2N \frac{e}{2(\pi + gN)} \varepsilon_{\mu\nu} \partial_\mu A_\nu(x) + \text{contact terms} . \quad (3.19)$$

This identity can be seen to hold if one evaluates the functional

$$\left\langle \exp \left(e \partial_\mu \left[j_{5\mu}(t) - 2N \frac{e}{2(\pi + gN)} \varepsilon_{\mu\nu} A_\nu(t) \right] \right) X \right\rangle . \quad (3.20)$$

Here X denotes an arbitrary monomial of the field operators and t is some test function that is used to smear the fields. The expression (3.20) can be computed with the methods developed below, and it turns out that it does not depend on t up to the announced contact term¹. Since this is true for arbitrary t and X , (3.19) holds. It has to be remarked that the coupling constant g for the Thirring term shows up in the anomaly equation. This is due to the fact that it is the $U(1)$ vector current that enters the Thirring term, leading to an extra contribution to the anomaly. Obviously when setting $g = 0$ the usual result is recovered.

As in QCD, the anomaly breaks the symmetry down to $SU(N)_L \times SU(N)_R \times U(1)_V$. Since the right hand side of the anomaly equation (3.19) is a divergence, the formal arguments that were applied in QCD to define a conserved current \tilde{j}_5 can be repeated. Thus when considering the symmetry properties, the toy model is adequate for studying the problematic aspects of the formulation of the $U(1)$ -problem.

For finite fermion masses the chiral symmetry is already broken at the classical level. As long as all masses are equal there is a remaining $U(N)_V$ symmetry, which is further reduced to $(U(1)_V)^N$ in the case of arbitrary masses.

3.3 Topologically nontrivial configurations for $U(1)$ gauge fields in two dimensions

In two dimensions also the gauge group $U(1)$ allows for topologically nontrivial configurations. There explicit calculations are much easier than for $SU(2)$. Since $U(1)$ gauge fields are relevant for the model under consideration, I decided to include the 2d discussion as an illustration of Section 2.1.

The topological index (Chern number) is given by

$$\nu[A] := \frac{e}{4\pi} \int d^2x \varepsilon_{\mu\nu} F_{\mu\nu}(x) = \frac{e}{2\pi} \int d^2x \partial_\mu K_\mu(x) , \quad (3.21)$$

where

$$K_\mu(x) := \varepsilon_{\mu\nu} A_\nu(x) . \quad (3.22)$$

¹Contact terms do not contribute when performing the Osterwalder-Schrader reconstruction [31].

Since $F_{\mu\nu}$ is gauge invariant, so is the index. For gauge field configurations with the boundary condition (this is sufficient for finite action (compare (2.1))

$$A_\mu(x) \xrightarrow{x^2 \rightarrow \infty} -\frac{i}{e} \left(\partial_\mu S(x) \right) \left(S(x) \right)^{-1}, \quad (3.23)$$

$\nu[A]$ can easily be seen to be an integer. Using Stokes theorem and choosing the integration boundary to be a circle gives

$$\nu[A] = \lim_{R \rightarrow \infty} \frac{e}{2\pi} \int_0^{2\pi} R d\varphi \hat{r}_\mu K_\mu = \lim_{R \rightarrow \infty} \frac{-i}{2\pi} \int_0^{2\pi} R d\varphi \hat{r}_\mu \varepsilon_{\mu\nu} (\partial_\nu S) (S)^{-1}. \quad (3.24)$$

In the last step the asymptotic form (3.23) was inserted. \hat{r} denotes the unit vector pointing in radial direction. Since $\hat{r}_\mu \varepsilon_{\mu\nu} = \hat{\varphi}_\nu$ and $\nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\varphi}_r \frac{1}{r} \frac{\partial}{\partial \varphi}$, one ends up with

$$\nu[A] = \lim_{R \rightarrow \infty} \frac{-i}{2\pi} \int_0^{2\pi} d\varphi \left(\frac{\partial}{\partial \varphi} S(R, \varphi) \right) \left(S(R, \varphi) \right)^{-1}. \quad (3.25)$$

The right hand side can be seen to be an integer by inserting the explicit parametrization $S(R, \varphi) = \exp(if(R, \varphi))$ where f is a continuous function.

The standard representative $A^{(n)}$ for a configuration with index $\nu[A^{(n)}] = n$ is given by

$$A_\mu^{(n)} = -\frac{i}{e} \frac{x^2}{x^2 + \lambda^2} \left(\partial_\mu S^{(n)}(x) \right) \left(S^{(n)}(x) \right)^{-1}, \quad (3.26)$$

with

$$S^{(n)}(x) = \left(S^{(1)}(x) \right)^n, \quad S^{(1)}(x) = \frac{x_2 - ix_1}{\sqrt{x_1^2 + x_2^2}}. \quad (3.27)$$

As in QCD, the construction of the θ -vacuum will be performed in temporal gauge ($A_2^{(n)} = 0$). In the discussion below I restrict myself to the configuration $A^{(1)}$ with winding number equal to 1. Inserting (3.27) in (3.26) then gives

$$A_1^{(1)}(x) = \frac{1}{e} \frac{-x_2}{x^2 + \lambda^2}, \quad A_2^{(1)}(x) = \frac{1}{e} \frac{x_1}{x^2 + \lambda^2}. \quad (3.28)$$

Going to temporal gauge means to gauge transform $A^{(1)}$

$$A_\mu^{(1)}(x) \longrightarrow \tilde{A}_\mu^{(1)}(x) = A_\mu^{(1)}(x) - \frac{i}{e} \left(\partial_\mu \tilde{S}(x) \right) \left(\tilde{S}(x) \right)^{-1}, \quad (3.29)$$

with the restriction $\tilde{A}_2^{(1)}(x) = 0$ which amounts to the partial differential equation

$$\partial_2 \tilde{S}(x) = -i \frac{x_1}{x^2 + \lambda^2} \tilde{S}(x). \quad (3.30)$$

It is solved by

$$\tilde{S}(x) = \exp \left(-i \frac{x_1}{\sqrt{x_1^2 + \lambda^2}} \left[\arctan \left(\frac{x_2}{\sqrt{x_1^2 + \lambda^2}} \right) - \pi \left(\frac{1}{2} + m \right) \right] \right), \quad (3.31)$$

where $m \in \mathbb{Z}$ comes about by a proper choice of the integration constant. The nonvanishing field component $\tilde{A}_1^{(1)}$ is obtained from (3.28) and (3.29)

$$\begin{aligned} \tilde{A}_1^{(1)}(x) = & -\frac{1}{e} \frac{x_2}{x_1^2 + x_2^2 + \lambda^2} \left[1 + \frac{x_1^2}{(x_1^2 + \lambda^2)^{\frac{3}{2}}} \right] \\ & - \frac{1}{e} \frac{\lambda^2}{(x_1^2 + \lambda^2)^{\frac{3}{2}}} \left[\arctan \left(\frac{x_2}{\sqrt{x_1^2 + \lambda^2}} \right) - \pi \left(\frac{1}{2} + m \right) \right]. \end{aligned} \quad (3.32)$$

Its asymptotic form can be written as

$$A_1^{(1)}(x) \longrightarrow \begin{cases} -\frac{i}{e} \left(\partial_1 S^{(m)}(x_1) \right) \left(S^{(m)}(x_1) \right)^{-1} & \text{for } x_2 \rightarrow -\infty \\ -\frac{i}{e} \left(\partial_1 S^{(m+1)}(x_1) \right) \left(S^{(m+1)}(x_1) \right)^{-1} & \text{for } x_2 \rightarrow +\infty, \end{cases} \quad (3.33)$$

with

$$S^{(l)}(x_1) = \left(S^{(1)}(x_1) \right)^l, \quad S^{(1)}(x_1) = \exp \left(i\pi \frac{x_1}{\sqrt{x_1^2 + \lambda^2}} \right). \quad (3.34)$$

Those two equations have to be compared with (2.8) and (2.9) for the 4d instanton. One easily reads off

$$\lim_{x_1 \rightarrow +\infty} S^{(l)}(x_1) = \lim_{x_1 \rightarrow -\infty} S^{(l)}(x_1) \quad \forall l, \quad (3.35)$$

and the (one dimensional) space can be compactified to S^1 . The winding number $\nu[A]$ can be written as $\nu[A] = \nu_+[A] - \nu_-[A]$, by choosing the integration boundary in (3.24) to be a rectangle which replaces the hypercylinder of Figure 2.1. The rest of the construction of the θ -vacua can be taken over from Section 2.1 immediately.

3.4 Outline of the construction

Vacuum expectation values of operators $P[\bar{\psi}, \psi, A, h]$ are formally defined as functional integrals

$$\langle P[\bar{\psi}, \psi, A, h] \rangle := \frac{1}{Z} \int \mathcal{D}h \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi P[\bar{\psi}, \psi, A, h] e^{-S[\bar{\psi}, \psi, A, h]}. \quad (3.36)$$

The normalization constant Z is chosen such that $\langle 1 \rangle = 1$. Obviously this is only a formal expression which has to be given some mathematical meaning.

As I will discuss in the next chapter it is not possible to find a simple expression of the fermion determinant for massive QED₂. Thus the strategy will be to expand the mass term $\exp(-S_M[\bar{\psi}, \psi])$ in a power series to obtain

$$\begin{aligned} \langle P[\bar{\psi}, \psi, A, h] \rangle &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{Z} \int \mathcal{D}h \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi P[\bar{\psi}, \psi, A, h] (S_M[\bar{\psi}, \psi])^n \\ &\times \exp \left(-S_G[A] - S_h[h] - S_F[\bar{\psi}, \psi, A, h] \right). \end{aligned} \quad (3.37)$$

Of course also the denominator has to be expanded (compare Equation (3.38) below). $P[\bar{\psi}, \psi, A, h]$ and hence also $P[\bar{\psi}, \psi, A, h](S_M[\bar{\psi}, \psi])^n$ are polynomials of the Grassmann variables with coefficients that are functionals of A_μ and h_μ . They can be generated taking derivatives of

$$\begin{aligned} F[\bar{\eta}, \eta, a, A, h] &:= \\ \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left(\sum_{b=1}^N \left\{ (\bar{\eta}^{(b)}, \psi^{(b)}) + (\bar{\psi}^{(b)}, \eta^{(b)}) + ie(a_\mu^{(b)}, j_\mu^{(b)}) \right\} \right) e^{-S_F[\bar{\psi}, \psi, A, h]}, \end{aligned} \quad (3.38)$$

with respect to the Grassmann sources $\bar{\eta}^{(b)}, \eta^{(b)}$ and the real number sources $a_\mu^{(b)}$. The latter are taken transverse, i.e. they obey $\partial_\mu a_\mu^{(b)} = 0$. They are convenient if one considers expectation values of vector currents. Since they couple in the same way as the gauge field they can be treated by replacing $A \rightarrow A + a^{(b)}$ in the fermion action. After the functional derivation the sources will be set to zero.

The generating functional $F[\bar{\eta}, \eta, a, A, h]$ can be expressed by means of the Berezin integral [11]

$$F[\bar{\eta}, \eta, a, A, h] = c \prod_{b=1}^N \det [1 - K(\mathcal{B}^{(b)})] \exp \left(\sum_{b=1}^N (\bar{\eta}^{(b)}, G(B^{(b)}) \eta^{(b)}) \right). \quad (3.39)$$

c denotes a constant that will be included in Z . The fermion determinant and the operator $K(\mathcal{B}^{(b)})$ will be discussed in detail in the next section. This expression shows that polynomials of Grassmann variables turn to new polynomials that depend on propagators G and functional derivatives $\frac{\delta}{\delta a}$ when the fermions are integrated out

$$Q_n[\bar{\psi}, \psi, A, h] := P[\bar{\psi}, \psi, A, h] (S_M[\bar{\psi}, \psi])^n \longrightarrow \tilde{Q}_n[G, \frac{\delta}{\delta a}, A, h]. \quad (3.40)$$

The propagator $G(x, y; B^{(b)})$ is the inverse kernel of the fermion action

$$G(x, y; B^{(b)}) := [\not{\partial} - i \not{B}^{(b)}]^{-1}(x, y), \quad (3.41)$$

where I defined

$$B_\mu^{(b)} := e\left(A_\mu + a_\mu^{(b)}\right) + \sqrt{g}h_\mu . \quad (3.42)$$

In two dimensions G was found by Schwinger [52]. The explicit derivation is given in Appendix A.1 and I only quote the result here

$$G(x, y; B) = G^o(x - y) e^{i[\Phi(x) - \Phi(y)]} , \quad (3.43)$$

where

$$\Phi(x) = - \int d^2z D(x - z) \left(\partial_\mu B_\mu(z) + i\gamma_5 \varepsilon_{\mu\nu} \partial_\mu B_\nu(z) \right) . \quad (3.44)$$

The free fermion propagator $G^o(x - y)$ can also be found in the appendix

$$G^o(x - y) = \frac{1}{2\pi} \frac{\gamma_\mu x_\mu}{x^2} . \quad (3.45)$$

Thus after having integrated out the fermions the final expression for expectation values reads

$$\langle P[\bar{\psi}, \psi, A, h] \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{Z} \left\langle \tilde{Q}_n \left[G, \frac{\delta}{\delta a}, A, h \right] e^{ie(a_\mu^{(b)}, j_\mu^{(b)})} \right\rangle \Big|_{a=0} . \quad (3.46)$$

The expectation values $\langle .. \rangle_0$ of the massless theory are defined as

$$\begin{aligned} \langle Q[\bar{\psi}, \psi, A, h] \rangle_0 &:= \\ \frac{1}{Z_0} \int \mathcal{D}h \mathcal{D}A Q'[G, \frac{\delta}{\delta a}, A, h] e^{-S_G[A] - S_h[h]} \prod_{b=1}^N \det \left[1 - K \left(\mathcal{B}^{(b)} \right) \right] \Big|_{a=0} . \end{aligned} \quad (3.47)$$

This expression can now be given a precise mathematical meaning. In the next section I will construct the fermion determinant $\det \left[1 - K \left(\mathcal{B}^{(b)} \right) \right]$ in the external field $B^{(b)}$. It will turn out that it is Gaussian in $B^{(b)}$. Together with $\frac{1}{Z_0} \mathcal{D}h \mathcal{D}A e^{-S_G[A] - S_h[h]}$ this will amount to Gaussian measures for the gauge and the auxiliary field that will be derived in the third section of the next chapter.

Note that the normalization constant Z of the massive theory also has an expansion in terms of expectation values $\langle .. \rangle_0$

$$Z = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle \tilde{S}_{M,n}[G] \rangle_0 . \quad (3.48)$$

Chapter 4

Construction of the massless model

The aim of this chapter is to give a precise mathematical meaning to the objects defined in the last section, such that one can start to evaluate expectation values of physical interest. In particular I will discuss the fermion determinant and construct measures for A_μ and h_μ . Furthermore the propagator will be simplified by rewriting it in terms of scalar fields φ and θ .

4.1 The fermion determinant

When outlining the general strategy in the last chapter, the fermion determinant¹ was only introduced formally when integrating out the fermions

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-(\bar{\psi}, Q\psi)} = \det[Q] . \quad (4.1)$$

This formula is inspired by reducing the system to finitely many degrees of freedom [11] (e.g. by putting the theory on a finite lattice). Nevertheless the right hand side has no mathematical meaning in the continuum yet.

The idea is to transform the problem by formal manipulations in such a way that one has to compute $\det[1 - K]$, where K is a trace class operator. In particular formally one obtains

$$\det[\not{D} - ie \not{B}] = \det[\not{D}] \det[1 - ie \not{B} \not{D}^{-1}] \quad (4.2)$$

and includes the infinite constant $\det[\not{D}]$ in the normalization constant Z (see (3.36)). This procedure was performed in a mathematically rigorous setting for QED₂ in a lattice regularization [16].

¹A very nice introduction to fermion determinants can be found in [55].

Once approached at $\det[1 - K]$ one has at hand the well known Fredholm determinant (see e.g. Vol. 4 of [49]), that is defined by

$$\det[1 - K] := \sum_{n=0}^{\infty} (-1)^n \text{Tr}[\Lambda^n(K)] . \quad (4.3)$$

$\Lambda^n(K)$ denotes the operator that is induced by K on the n -fold antisymmetric tensor product

$$\Lambda^n(K) \varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_n := K\varphi_1 \wedge K\varphi_2 \wedge \dots \wedge K\varphi_n . \quad (4.4)$$

(4.3) can be shown [62] to have the expansion

$$\det[1 - K] = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[K^n] \right) , \quad (4.5)$$

which converges for $\text{Tr}|K| < 1$. I will have to consider operators K which are not trace class, but a suitable power is. This motivates the consideration of trace ideals \mathcal{I}_q , $q \geq 1$ defined as (see Vol. 2 of [49])

$$\mathcal{I}_q := \left\{ \text{compact operators } C \mid |C|^q := (C^*C)^{\frac{q}{2}} \text{ is trace class} \right\} , \quad (4.6)$$

with the norms

$$\|C\|_q := \left(\text{Tr}|C|^q \right)^{\frac{1}{q}} . \quad (4.7)$$

From (4.5) there follows a natural definition of modified determinants adapted to the trace ideals. Let $K \in \mathcal{I}_q$. Then

$$\det_q[1 - K] := \exp \left(- \sum_{n=q}^{\infty} \frac{1}{n} \text{Tr}[K^n] \right) . \quad (4.8)$$

To understand from a physical point of view what has been done, I go back to QED₂ (see [54]). After a similarity transformation

$$K(\mathcal{B}) := \not{P} |P|^{-\frac{3}{2}} \not{B} |P|^{-\frac{1}{2}} , \quad \text{with } P_\mu := -i\partial_\mu , \quad (4.9)$$

considered as an operator on two component, square integrable functions on \mathbb{R}^2 . For the external field B see equation (3.42). If

$$\int d^2x |B_\mu|^q < \infty , \quad \forall \quad q \geq \frac{1}{2} , \quad (4.10)$$

(i.e. $|B_\mu|^{\frac{1}{2}} \in \bigcap_{q \geq 1} L^q$), then $K(\mathcal{B}) \in \mathcal{I}_q$, for all $q > 2$ (see [57]). This requirement implies the inclusion of a cutoff for the fields entering B . For the gauge field A_μ this cutoff can entirely be removed in the end (see [56] for a discussion), and thus is not explicitly quoted here. For the auxiliary field h_μ

this is not the case and the cutoff procedure will be made explicit where it is necessary (see (5.23)).

From (4.10) there follows that the modified determinant under consideration is

$$\det_3[1 - K(\mathcal{B})] := \exp \left(- \sum_{n=3}^{\infty} \frac{1}{n} \text{Tr}[K(\mathcal{B})^n] \right). \quad (4.11)$$

It differs from Expression (4.5) by the absence of $-\text{Tr}[K(\mathcal{B})]$ and $-\frac{1}{2}\text{Tr}[K(\mathcal{B})^2]$ in the exponent. The first term is zero if it is renormalized properly in accordance with Furry's theorem (see below). The second one corresponds to the diagram of Figure 4.1, which is the only divergent graph in two dimensions.

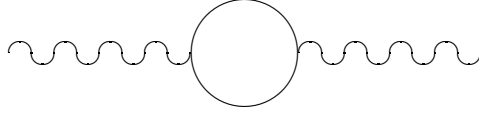


Figure 4.1 : The diagram corresponding to $\text{Tr}[K(\mathcal{B})^2]$.

It is well known how to renormalize this diagram in a gauge invariant way. To obtain a legitimate expression for a *renormalized determinant*, the renormalized trace has to be restored. Thus the final definition of the renormalized determinant reads

$$\det_{ren}[1 - K(\mathcal{B})] = \det_3[1 - K(\mathcal{B})] \exp \left(- \frac{1}{2} \text{Tr}_{ren}[K^2(\mathcal{B})] \right). \quad (4.12)$$

First I discuss $\det_3[1 - K(\mathcal{B})]$. The following lemma holds:

Lemma 4.1 :

For $|B_\mu|^{1/2} \in \bigcap_{q \geq 1} L^q$, $\partial_\mu B_\mu = 0$

$$\det_3[1 - K(\mathcal{B})] = 1. \quad (4.13)$$

Proof:

Making use of Furry's [38] theorem one finds that the determinant is even in e, \sqrt{g} . In particular under a charge conjugation \mathcal{C} (h_μ transforms in the same way A_μ does)

$$K(\mathcal{B}) \xrightarrow{\mathcal{C}} -K(\mathcal{B}). \quad (4.14)$$

Since \mathcal{C} is a symmetry of the model only even terms have to be taken into account, and one obtains

$$\det_3[1 - K(\mathcal{B})] = \exp \left(- \sum_{n=2}^{\infty} \text{Tr}[K^{2n}(\mathcal{B})] \right). \quad (4.15)$$

From the last equation and Definition (4.9) for $K(\mathcal{B})$ there follows immediately

$$\det_3[1 - K(\mathcal{B})] = \det_3[1 - K(\gamma_5 \mathcal{B})] . \quad (4.16)$$

Define $\tilde{\mathcal{B}} := \gamma_5 \mathcal{B}$ which explicetely gives $\tilde{B}_1 = -iB_2 = i\partial_1 f$ and $\tilde{B}_2 = iB_1 = i\partial_2 f$, showing that \tilde{B}_μ is a pure gauge. I already used the transversality of B_μ , that allows to write $B_\mu = \varepsilon_{\mu\nu}\partial_\nu f$ for some scalar function f . From the definition it follows that the determinant is invariant under the gauge transformation $\psi(x) \rightarrow \exp(if(x))\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)\exp(-if(x))$ which removes \tilde{B}_μ . Now one can finish the proof

$$\det_3[1 - K(\mathcal{B})] = \det_3[1 - K(\tilde{\mathcal{B}})] = \det_3[1 - K(0)] = 1 . \quad \square \quad (4.17)$$

$\text{Tr}_{ren}(K^2(\mathcal{B}))$ can be evaluated easily by using e.g. dimensional regularization. To handle the infrared problem, I work with finite mass m and perform the massless limit in the end.

$$\begin{aligned} \text{Tr}_{ren}(K^2(\mathcal{B})) &= \text{Tr}_\gamma \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} d^2 x d^2 y e^{-ipx} K(\mathcal{B})(x) e^{iqx} e^{-iqy} K(\mathcal{B})(y) e^{ipy} \\ &= \int d^2 k \hat{B}_\mu(k) \hat{B}_\nu(-k) \hat{T}_{\mu\nu}(k) . \end{aligned} \quad (4.18)$$

\hat{B}_μ is the Fourier transform of B_μ , and $\hat{T}_{\mu\nu}$ is given by (now the dimensional regularization comes in; $\omega := 1 - \epsilon$)

$$\begin{aligned} \hat{T}_{\mu\nu}(k) &= \int \frac{d^{2\omega} p}{(2\pi)^{2\omega}} \frac{\text{Tr}(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta)(p_\alpha - k_\alpha)p_\beta - m^2 \text{Tr}(\gamma_\mu \gamma_\nu)}{[(p - k)^2 + m^2][p^2 + m^2]} \\ &= \int_{x=0}^1 dx \int \frac{d^{2\omega} r}{(2\pi)^{2\omega}} \frac{\text{Tr}(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta)(r_\alpha - xk_\alpha)(r_\beta + (1-x)k_\beta) - m^2 2\delta_{\mu\nu}}{[r^2 + m^2 + k^2 x(1-x)]^2} \\ &= \text{Tr}(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta) \int_{x=0}^1 dx \frac{1}{(4\pi)^\omega \Gamma(2)} \left(\frac{1}{2} \delta_{\alpha\beta} \frac{\Gamma(1-\omega)}{[m^2 + k^2 x(1-x)]^{1-\omega}} \right. \\ &\quad \left. - x(x-1)k_\alpha k_\beta \frac{\Gamma(2-\omega)}{[m^2 + k^2 x(1-x)]^{2-\omega}} \right) \\ &\quad - 2\delta_{\mu\nu} m^2 \int_{x=0}^1 dx \frac{1}{(4\pi)^\omega \Gamma(2)} \frac{\Gamma(2-\omega)}{[m^2 + k^2 x(1-x)]^{2-\omega}} . \end{aligned} \quad (4.19)$$

In the first step a Feynman parameter x was introduced and a variable transformation $r := p - k(1-x)$ was performed to bring the integral to standard form where it can be solved using well known dimensional regularization formulas (see eg. [47]). In $2\omega = 2(1 - \epsilon)$ dimensions one has the following trace identities

$$\text{Tr}(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta) k_\alpha k_\beta = 2(2k_\mu k_\nu - \delta_{\mu\nu} k^2) + O(\epsilon) ,$$

$$\text{Tr}(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta) \delta_{\alpha\beta} = 4\epsilon \delta_{\mu\nu} + O(\epsilon^2). \quad (4.20)$$

The Feynman parameter integrals over x can also be expanded in ϵ and then be solved. Putting things together one ends up with

$$\begin{aligned} \hat{T}_{\mu\nu}(k) &= \frac{1}{2\pi} \delta_{\mu\nu} + \frac{1}{2\pi} \left(2 \frac{k_\mu k_\nu}{k^2} - \delta_{\mu\nu} \right) \left(-1 + \left(\sqrt{\frac{m^2}{k^2} + \frac{1}{4}} - \frac{1}{4} \frac{1}{\sqrt{\frac{m^2}{k^2} + \frac{1}{4}}} \right) l(k, m) \right) \\ &\quad - \frac{1}{2\pi} \delta_{\mu\nu} \frac{m^2}{k^2} \frac{1}{\sqrt{\frac{m^2}{k^2} + \frac{1}{4}}} l(k, m) + O(\epsilon), \end{aligned} \quad (4.21)$$

where

$$l(k, m) = \ln \left(\frac{\sqrt{\frac{m^2}{k^2} + \frac{1}{4}} + \frac{1}{2}}{\sqrt{\frac{m^2}{k^2} + \frac{1}{4}} - \frac{1}{2}} \right). \quad (4.22)$$

It is a remarkable feature of QED₂ that the terms of order $1/\epsilon$ cancel and no counterterms have to be added. QED₂ therefore is a *finite theory*.

Finally I perform the limit $m \rightarrow 0$ to obtain

$$\hat{T}_{\mu\nu}(k) = \frac{1}{\pi} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \left(1 + 2 \frac{m^2}{k^2} \ln \left(\frac{m^2}{k^2} \right) + O \left(\frac{m^4}{k^4} \ln \left(\frac{m^2}{k^2} \right) \right) \right). \quad (4.23)$$

In the massless limit the integral kernel $\hat{T}_{\mu\nu}(k)$ reduces to the transversal projector already encountered in (3.10)

$$T_{\mu\nu} := \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \quad (4.24)$$

Using (4.12) and Lemma 4.1 one ends up with a remarkably simple expression for the fermion determinant in the massless case

$$\det_{\text{ren}}[1 - K(\mathcal{B})] = e^{-\frac{1}{2\pi} \|B^T\|_2^2}, \quad B_\mu^T := T_{\mu\nu} B_\nu. \quad (4.25)$$

It has to be remarked that the condition (4.10) on B_μ and thus on A_μ (for $g = 0$) implies zero winding. This fact will leave a trace in the properties of the expectation functional which will be discussed in the next chapter.

4.2 Remarks on the massive case

In the last section the following small m behaviour for the momentum space kernel $\hat{T}_{\mu\nu}(k)$ of $\text{Tr}_{\text{ren}}[K^2(\mathcal{B})]$ has been established

$$\hat{T}_{\mu\nu}(k) = \frac{1}{\pi} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \left(1 + 2 \frac{m^2}{k^2} \ln \left(\frac{m^2}{k^2} \right) + O \left(\frac{m^4}{k^4} \ln \left(\frac{m^2}{k^2} \right) \right) \right). \quad (4.26)$$

One can try to find a similar expansion in m for the traces $\text{Tr}[K^{2n}(\mathcal{B})]$, $n > 1$ that build up $\det_3[1 - K(\mathcal{B})]$, which differs from 1 in the case of finite mass. If one could show that those higher traces decrease faster than $\frac{m^2}{k^2} \ln(\frac{m^2}{k^2})$, it would make sense to perform a mass expansion of the fermion determinant directly.

As a matter of fact this optimistic scenario does not hold. By discussing $\text{Tr}[K^4(\mathcal{B})]$ I will make it plausible that all $\text{Tr}[K^{2n}(\mathcal{B})]$, $n > 1$ do not vanish faster than $\frac{m^2}{k^2} \ln(\frac{m^2}{k^2})$.

Using gauge invariance and the fact that $\gamma_5 \mathcal{B}$ is a pure gauge one obtains

$$\begin{aligned} 0 &= \text{Tr}[K(\gamma_5 \mathcal{B})^4] \\ &= \text{Tr}\left[\frac{-\not{\partial} + m}{-\Delta + m^2} \mathcal{B} \frac{-\not{\partial} - m}{-\Delta + m^2} \mathcal{B} \frac{-\not{\partial} + m}{-\Delta + m^2} \mathcal{B} \frac{-\not{\partial} - m}{-\Delta + m^2} \mathcal{B}\right]. \end{aligned} \quad (4.27)$$

In the last step $[\gamma_5, \mathcal{B}] = 0$ and $\gamma_5^2 = 1$ were used. This allows to write

$$\begin{aligned} \text{Tr}[K(\mathcal{B})^4] &= \text{Tr}[K(\mathcal{B})^4] - \text{Tr}[K(\gamma_5 \mathcal{B})^4] \\ &= 2m \text{Tr}\left[\frac{-\not{\partial} + m}{-\Delta + m^2} \mathcal{B} \frac{1}{-\Delta + m^2} \mathcal{B} \frac{-\not{\partial} + m}{-\Delta + m^2} \mathcal{B} \frac{-\not{\partial} - m}{-\Delta + m^2} \mathcal{B}\right] \\ &\quad + 2m \text{Tr}\left[\frac{-\not{\partial} + m}{-\Delta + m^2} \mathcal{B} \frac{-\not{\partial} + m}{-\Delta + m^2} \mathcal{B} \frac{-\not{\partial} + m}{-\Delta + m^2} \mathcal{B} \frac{1}{-\Delta + m^2} \mathcal{B}\right] \\ &= 8m^2 \text{Tr}\left[\frac{1}{-\Delta + m^2} \mathcal{B} \frac{1}{-\Delta + m^2} \mathcal{B} \frac{1}{-\Delta + m^2} \not{\partial} \mathcal{B} \frac{1}{-\Delta + m^2} \not{\partial} \mathcal{B}\right], \end{aligned} \quad (4.28)$$

where successively $\not{\partial}$ terms in the numerator were canceled. For the gauge field I make the following ansatz that trivially obeys the $\partial_\mu B_\mu = 0$ gauge condition

$$\hat{B}_1(q) = -q_2 f(q) \quad , \quad \hat{B}_2(q) = q_1 f(q) \quad , \quad (4.29)$$

where $f(q)$ is a $L^2(\mathbb{R})$ scalar function. Inserting this ansatz and Fourier transforming the trace one obtains

$$\begin{aligned} \text{Tr}[K(\mathcal{B})^4] &= \frac{16m^2}{(2\pi)^8} \int d^2 q_2 d^2 q_3 d^2 q_4 f(q_2 + q_3 + q_4) f(-q_2) f(-q_3) f(-q_4) \times \\ &\quad \int d^2 k \frac{-(q_2 + q_3 + q_4) \cdot q_2 q_3^2 q_4^2}{(k^2 + m^2)((k + q_2)^2 + m^2)((k + q_2 + q_3)^2 + m^2)((k + q_2 + q_3 + q_4)^2 + m^2)} . \end{aligned} \quad (4.30)$$

For non-exceptional momenta q_2, q_3, q_4 the k -integral behaves as $\ln(m)$ and one ends up with

$$\text{Tr}[K(\not{B})^4] \propto m^2 \ln(m) . \quad (4.31)$$

In the light of this result it is quite unlikely that $\text{Tr}[K^{2n}(\not{B})]$, $n > 2$ vanish faster than $m^2 \ln(m)$. Hence it is not very promising to try a mass expansion of the fermion determinant directly.

4.3 Measures for A and h

As announced the determinant comes out Gaussian in the external fields. In this section I will construct a common Gaussian measure for the formal expression

$$\frac{1}{Z_0} \mathcal{D}h \mathcal{D}A e^{-S_G[A] - S_h[h]} \prod_{b=1}^N \det_{ren} [1 - K(\not{B}^{(b)})] . \quad (4.32)$$

The field combination B_μ is taken from (3.42)

$$B_\mu^{(b)} := e(A_\mu + a_\mu^{(b)}) + \sqrt{g} h_\mu . \quad (4.33)$$

Inserting this into the result for the determinant (4.24) one obtains

$$\begin{aligned} \prod_{b=1}^N \det_{ren} [1 - K(\not{B}^{(b)})] &= \exp \left(-\frac{1}{2\pi} \sum_{b=1}^N \|B^{(b)T}\|_2^2 \right) = \\ &\exp \left(-\frac{e^2 N}{2\pi} (A, TA) - \frac{e\sqrt{g}N}{\pi} (h, TA) - \frac{gN}{2\pi} (h, Th) - \right. \\ &\left. \frac{e^2}{\pi} \left(A, T \sum_{b=1}^N a^{(b)} \right) - \frac{e\sqrt{g}}{\pi} \left(h, T \sum_{b=1}^N a^{(b)} \right) - \frac{e^2}{2\pi} \sum_{b=1}^N (a^{(b)}, Ta^{(b)}) \right) . \end{aligned} \quad (4.34)$$

Here the transverse projector T (3.10) was used to obtain $\|B^T\|_2^2 = (TB, TB) = (B, TB)$.

The gauge field action as well as the action for the auxiliary field can be written as quadratic forms (compare (3.2), (3.3))

$$\begin{aligned} S_G[A] &= \frac{1}{2} (A_\mu, [(1 - \lambda)\partial_\mu \partial_\nu - \triangle \delta_{\mu\nu}] A_\nu) , \\ S_h[h] &= \frac{1}{2} (h_\mu, [\delta_{\mu\nu} - \lambda' \partial_\mu \partial_\nu] h_\nu) . \end{aligned} \quad (4.35)$$

In a first step I unify the auxiliary field action and the quadratic term in h from the determinant to one common quadratic form

$$\frac{1}{2} (h, C^{-1}h) := S_h[h] + \frac{gN}{2\pi} (h, Th) . \quad (4.36)$$

One obtains

$$C_{\mu\nu}^{-1} = \delta_{\mu\nu} - \lambda' \partial_\mu \partial_\nu + \frac{gN}{\pi} \left(\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Delta} \right). \quad (4.37)$$

It is easy to verify (by Fourier transformation) that C^{-1} is a positive, nondegenerate operator in $\mathcal{S}^2(\mathbb{R}^2)$ and hence gives rise to a covariance operator C , which is given by

$$C_{\mu\nu} = \left(1 - \lambda' \Delta\right)^{-1} \delta_{\mu\nu} - \frac{\pi}{\pi + gN} \left(1 - \lambda' \Delta\right)^{-1} \left(\lambda' \Delta + \frac{gN}{\pi} \right) T_{\mu\nu}, \quad (4.38)$$

and reduces in the transverse limit to

$$\lim_{\lambda' \rightarrow \infty} C_{\mu\nu} = \frac{\pi}{\pi + gN} T_{\mu\nu}. \quad (4.39)$$

This is a proper covariance in the transverse subspace of $\mathcal{S}^2(\mathbb{R}^2)$. In the following I will only need this expression.

By completing the square one can include the mixed term $\frac{e\sqrt{g}N}{\pi}(h, TA)$

$$\begin{aligned} S_h[h] + \frac{gN}{2\pi}(h, Th) + \frac{e\sqrt{g}N}{\pi}(h, TA) &= \frac{1}{2}(h, C^{-1}h) + \frac{e\sqrt{g}N}{\pi}(h, TA) \\ &= \frac{1}{2}(h', C^{-1}h') - \frac{1}{2} \frac{ge^2N^2}{\pi^2}(A, TCA), \end{aligned} \quad (4.40)$$

where

$$h' = h + \frac{e\sqrt{g}N}{\pi} TCA = h + \frac{e\sqrt{g}N}{\pi + gN} TA. \quad (4.41)$$

The coordinate transform $h \rightarrow h'$ can be performed without any trouble in the Gaussian integrals and the observables and propagators can be rewritten in terms of h' .

There are now three pieces quadratic in A . One from the completion of the square (4.40), one from the determinant (4.34) and one from the action (4.35). They can be unified to one common quadratic form

$$\frac{1}{2}(A, Q^{-1}A) := S_G[A] + \frac{e^2N}{2\pi}(A, TA) - \frac{1}{2} \frac{ge^2N^2}{\pi^2}(A, TCA), \quad (4.42)$$

with

$$\begin{aligned} Q_{\mu\nu}^{-1} &= \left((1 - \lambda) \partial_\mu \partial_\nu - \Delta \delta_{\mu\nu} \right) + \frac{e^2N}{\pi} T_{\mu\nu} - \frac{ge^2N^2}{\pi^2} \frac{\pi}{\pi + gN} T_{\mu\nu} \\ &= \left(-\Delta + \frac{e^2N}{\pi + gN} \right) T_{\mu\nu} - \lambda \partial_\mu \partial_\nu. \end{aligned} \quad (4.43)$$

Again it is easy to check that Q^{-1} gives rise to a proper covariance Q on $\mathcal{S}^2(\mathbb{R})$. Q is given by

$$Q_{\mu\nu} = -\frac{1}{\lambda} \frac{\partial_\mu \partial_\nu}{\Delta} + \left(-\Delta + \frac{e^2 N}{\pi + gN} \right)^{-1} T_{\mu\nu} , \quad (4.44)$$

and reduces in the transverse subspace to

$$Q_{\mu\nu} = \left(-\Delta + \frac{e^2 N}{\pi + gN} \right)^{-1} T_{\mu\nu} . \quad (4.45)$$

The fact that the gauge field has a massive propagator is known as the *Schwinger mechanism*.

Finally I rewrite the terms in the determinant (4.34) where the sources $a^{(b)}$ and the fields h and A mix in terms of h' and A

$$\begin{aligned} & \frac{e^2}{\pi} \left(A, T \sum_{b=1}^N a^{(b)} \right) + \frac{e\sqrt{g}}{\pi} \left(h, T \sum_{b=1}^N a^{(b)} \right) \\ &= \frac{e^2}{\pi + gN} \left(A, T \sum_{b=1}^N a^{(b)} \right) + \frac{e\sqrt{g}}{\pi} \left(h', T \sum_{b=1}^N a^{(b)} \right) . \end{aligned} \quad (4.46)$$

Thus one has established that the initial expression (4.32) can be replaced by

$$\begin{aligned} & \frac{1}{Z_0} \mathcal{D}h \mathcal{D}A \exp \left(-\frac{1}{2} \left(A, Q^{-1} A \right) - \frac{1}{2} \left(h', C^{-1} h' \right) \right) \\ & \times \exp \left(-\frac{e^2}{\pi + gN} \left(A, T \sum_{b=1}^N a^{(b)} \right) - \frac{e\sqrt{g}}{\pi} \left(h', T \sum_{b=1}^N a^{(b)} \right) - \frac{e^2}{2\pi} \sum_{b=1}^N \left(a^{(b)}, T a^{(b)} \right) \right) , \end{aligned} \quad (4.47)$$

which has the clear meaning

$$\begin{aligned} & d\mu_Q[A] d\mu_C[h'] \\ & \times \exp \left(-\frac{e^2}{\pi + gN} \left(A, T \sum_{b=1}^N a^{(b)} \right) - \frac{e\sqrt{g}}{\pi} \left(h', T \sum_{b=1}^N a^{(b)} \right) - \frac{e^2}{2\pi} \sum_{b=1}^N \left(a^{(b)}, T a^{(b)} \right) \right) , \end{aligned} \quad (4.48)$$

in terms of Gaussian measures² $d\mu_Q[A]$ and $d\mu_C[h']$ with covariances Q and C given by (4.45) and (4.39). The poorly defined vacuum expectation value (3.47) of the massless model now has the precise definition

$$\begin{aligned} & \left\langle Q[\bar{\psi}, \psi, A, h] \right\rangle_0 := \int d\mu_Q[A] d\mu_C[h'] \quad Q' \left[G, \frac{\delta}{\delta a}, A, h' \right] \\ & \exp \left(-\frac{e^2}{\pi + gN} \left(A, T \sum_{b=1}^N a^{(b)} \right) - \frac{e\sqrt{g}}{\pi} \left(h', T \sum_{b=1}^N a^{(b)} \right) - \frac{e^2}{2\pi} \sum_{b=1}^N \left(a^{(b)}, T a^{(b)} \right) \right) \Big|_{a^{(b)}=0} . \end{aligned} \quad (4.49)$$

²For a short introduction to Gaussian measures see Appendix A.2

4.4 The propagator and the fields φ and θ

The fermion propagator in the transverse external field $B^{(b)}$ was obtained Appendix A.1 and reads (c.f. (A.15))

$$G(x, y; B^{(b)}) = \frac{1}{2\pi} \frac{1}{(x-y)^2} \begin{pmatrix} 0 & e^{-[\chi^{(b)}(x) - \chi^{(b)}(y)]} \overline{(\tilde{x} - \tilde{y})} \\ e^{+[\chi^{(b)}(x) - \chi^{(b)}(y)]} (\tilde{x} - \tilde{y}) & 0 \end{pmatrix}, \quad (4.50)$$

where

$$\chi^{(b)}(x) := \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} B_\nu^{(b)}(x) \quad , \quad \tilde{x} := x_1 + ix_2. \quad (4.51)$$

The external field $B^{(b)}$ that enters the propagator was defined in (3.42). It has to be rewritten in terms of h' . $\frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta}$ acts on it to give $\chi^{(b)}$, which reads in terms of h'

$$\chi^{(b)}(x) = \frac{e\pi}{\pi + gN} \varphi(x) + \sqrt{g} \theta(x) + e \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} a_\nu^{(b)}(x), \quad (4.52)$$

where I defined the scalar fields

$$\varphi(x) := \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} A_\nu(x) \quad , \quad \theta(x) := \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} h'_\nu(x). \quad (4.53)$$

It will turn out that it is more convenient to work with the scalar fields φ and θ . Obviously their measures are also Gaussian and the corresponding covariance operators \tilde{Q} and \tilde{C} are immediately obtained from Q and C

$$\tilde{Q} = -\frac{\varepsilon_{\rho\mu} \partial_\rho}{\Delta} Q_{\mu\nu} \frac{\varepsilon_{\sigma\nu} \partial_\sigma}{\Delta} = \frac{1}{-\Delta + \frac{e^2 N}{\pi + gN}} \frac{-1}{\Delta}, \quad (4.54)$$

and

$$\tilde{C} = -\frac{\varepsilon_{\rho\mu} \partial_\rho}{\Delta} C_{\mu\nu} \frac{\varepsilon_{\sigma\nu} \partial_\sigma}{\Delta} = \frac{\pi}{\pi + gN} \frac{-1}{\Delta}. \quad (4.55)$$

\tilde{C} behaves $\propto 1/p^2$ in momentum space which causes an ultraviolet problem. As was discussed above (see page 28) the construction of the determinant requires a cutoff. Here I adopt the following procedure. The scalar field $\chi(x)$ at the single space-time point x will be replaced by the convolute

$$\chi(x) \longrightarrow \int d^2\xi \chi(\xi) \delta_n(\xi - x) =: (\chi, \delta_n(x)), \quad (4.56)$$

where $\delta_n(x)$ denotes a δ -sequence peaked at x

$$\delta_n(\xi - x) := \int \frac{d^2p}{(2\pi)^2} e^{-\frac{|p|^2}{n}} e^{ip(\xi - x)}. \quad (4.57)$$

Thus the propagator takes the form

$$G(x, y; \chi) = \frac{1}{2\pi} \frac{1}{(x - y)^2} \begin{pmatrix} 0 & e^{-(\chi, \delta_n(x) - \delta_n(y))} \overline{(\tilde{x} - \tilde{y})} \\ e^{+(\chi, \delta_n(x) - \delta_n(y))} (\tilde{x} - \tilde{y}) & 0 \end{pmatrix}. \quad (4.58)$$

When one considers the limit $n \rightarrow \infty$, some of the operators will have to be multiplied with a wave function renormalization constant (see (5.53)) diverging as $n \rightarrow \infty$.

Chapter 5

Decomposition into clustering states and the vacuum angle

It will turn out that for vanishing fermion masses a certain class of operators violates the cluster decomposition property when using the expectation functional constructed so far. I am going to classify those operators and discuss their symmetry properties. Finally the expectation functional will be decomposed into clustering θ -vacua giving rise to proper states.

5.1 Clustering and the uniqueness of the vacuum

If there is any truth in the θ -vacuum philosophy, then one should face some problems with the vacuum state. The reason for this is that in the construction of the fermion determinant and the propagator it was assumed that the gauge fields decrease appropriately (compare (4.10) and Appendix A.1), and thus have zero winding. In the θ -language this means that up to now only transitions of the $\langle n|n\rangle$ type are included, whereas topologically nontrivial configurations enforce $\langle n + \nu|n\rangle$ (compare (2.14)) contributions as well. Thus one has to expect that the expectation functional constructed so far does not lead to a unique vacuum when the Osterwalder-Schrader reconstruction (see [31]) is performed. The ergodicity axiom which guarantees the uniqueness of the vacuum state is not fulfilled. Ergodicity is equivalent to the cluster property (compare [31], Section 19.7)

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^t d\tau \left[\langle AT(\tau)B \rangle - \langle A \rangle \langle B \rangle \right] \stackrel{!}{=} 0 . \quad (5.1)$$

$T(\tau)$ denotes time translation, and A, B are arbitrary polynomials in the fields smeared with test functions¹. To detect expectation values that violate clustering, and thus connect different vacua, I will consider a rather general ansatz for A and B in the next section.

It is more convenient to consider the $\tau \rightarrow \infty$ limit of

$$C(\tau) := \langle AT(\tau)B \rangle - \langle A \rangle \langle B \rangle . \quad (5.2)$$

Obviously a nonvanishing limit of $C(\tau)$ implies the violation of clustering in the formulation (5.1).

5.2 Identification of operators that violate clustering

To identify the operators that violate clustering, I start with an ansatz containing only the chiral densities $\bar{\psi}^{(b)} P_{\pm} \psi^{(b)}$, ($P_{\pm} := (1 \pm \gamma_5)/2$) and discuss the effect of adding vector currents and other modifications later. Define

$$C(\tau) := C_1(\tau) - C_2 \quad (5.3)$$

where

$$\begin{aligned} C_1(\tau) := & \left\langle \prod_{b=1}^N \prod_{i=1}^{n_b} \bar{\psi}^{(b)}(x_i^{(b)} + \hat{\tau}) P_+ \psi^{(b)}(x_i^{(b)} + \hat{\tau}) \prod_{i=1}^{m_b} \bar{\psi}^{(b)}(y_i^{(b)} + \hat{\tau}) P_- \psi^{(b)}(y_i^{(b)} + \hat{\tau}) \right. \\ & \times \left. \prod_{i=1}^{n'_b} \bar{\psi}^{(b)}(x'_i{}^{(b)}) P_+ \psi^{(b)}(x'_i{}^{(b)}) \prod_{i=1}^{m'_b} \bar{\psi}^{(b)}(y'_i{}^{(b)}) P_- \psi^{(b)}(y'_i{}^{(b)}) \right\rangle_0 \end{aligned} \quad (5.4)$$

and

$$\begin{aligned} C_2 := & \left\langle \prod_{b=1}^N \prod_{i=1}^{n_b} \bar{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{i=1}^{m_b} \bar{\psi}^{(b)}(y_i^{(b)}) P_- \psi^{(b)}(y_i^{(b)}) \right\rangle_0 \\ & \times \left\langle \prod_{b=1}^N \prod_{i=1}^{n'_b} \bar{\psi}^{(b)}(x'_i{}^{(b)}) P_+ \psi^{(b)}(x'_i{}^{(b)}) \prod_{i=1}^{m'_b} \bar{\psi}^{(b)}(y'_i{}^{(b)}) P_- \psi^{(b)}(y'_i{}^{(b)}) \right\rangle_0 . \end{aligned} \quad (5.5)$$

$\hat{\tau}$ denotes the vector of length τ in 2-direction. Violation of the cluster property now manifests itself in a nonvanishing limit

$$\lim_{\tau \rightarrow \infty} C(\tau) =: C \neq 0 . \quad (5.6)$$

It will be obtained for certain n_b, m_b, n'_b, m'_b .

¹In fact A, B can be taken from \mathcal{E} which is the closure of the vectors $\sum_j c_j \exp(\phi(f_j))$, $c_j \in \mathbb{C}$, $f_j \in \mathcal{D}$, in the $L_2(d\mu[\phi])$ inner product.

Since it is rather hopeless to evaluate the traces over general products of γ -matrices, I make use of the special form (4.58) of the propagator

$$G(x, y; \chi) = \frac{1}{2\pi} \frac{1}{(x-y)^2} \begin{pmatrix} 0 & e^{-(\chi, \delta_n(x) - \delta_n(y))} (\tilde{x} - \tilde{y}) \\ e^{+(\chi, \delta_n(x) - \delta_n(y))} (\tilde{x} - \tilde{y}) & 0 \end{pmatrix}. \quad (5.7)$$

With the chosen representation of the γ -algebra it has only off diagonal entries. For the evaluation of $C_1(\tau)$ and C_2 no sources $a^{(b)}$ are necessary and $\chi(x)$ (see (4.52)) reduces to

$$\chi(x) = \frac{e \pi}{\pi + gN} \varphi(x) + \sqrt{g} \theta(x). \quad (5.8)$$

Due to the chosen repräsentation of γ_5 the chiral densities are given by

$$\bar{\psi}^{(b)} P_+ \psi^{(b)} = \bar{\psi}_1^{(b)} \psi_1^{(b)} \quad , \quad \bar{\psi}^{(b)} P_- \psi^{(b)} = \bar{\psi}_2^{(b)} \psi_2^{(b)}. \quad (5.9)$$

The propagator (5.7) implies that $\bar{\psi}^{(b)} P_+ \psi^{(b)}$ and $\bar{\psi}^{(b)} P_- \psi^{(b)}$ have to come in pairs for all flavors b in order to allow complete contractions of the fermions which is necessary for nonvanishing results. Thus $C_1(\tau)$ does not vanish only for

$$n_b + n'_b = m_b + m'_b \quad , \quad b = 1, \dots, N. \quad (5.10)$$

After some reordering $C_1(\tau)$ reads

$$\begin{aligned} s \left\langle \prod_{b=1}^N \prod_{i=1}^{n_b} \psi_1^{(b)}(x_i^{(b)} + \hat{\tau}) \prod_{i=1}^{m'_b} \bar{\psi}_2^{(b)}(y_i'^{(b)}) \prod_{i=1}^{n'_b} \psi_1^{(b)}(x_i'^{(b)}) \prod_{i=1}^{m_b} \bar{\psi}_2^{(b)}(y_i^{(b)} + \hat{\tau}) \right. \\ \left. \times \prod_{i=1}^{m'_b} \psi_2^{(b)}(y_i'^{(b)}) \prod_{i=1}^{n_b} \bar{\psi}_1^{(b)}(x_i^{(b)} + \hat{\tau}) \prod_{i=1}^{m_b} \psi_2^{(b)}(y_i^{(b)} + \hat{\tau}) \prod_{i=1}^{n'_b} \bar{\psi}_1^{(b)}(x_i'^{(b)}) \right\rangle_0. \end{aligned} \quad (5.11)$$

s denotes an overall sign depending on n_b, m_b, n'_b, m'_b which is not relevant for the following.

One can make use of the simple exponential dependence of the propagator on the external fields to factorize $C_1(\tau)$. Whenever $\bar{\psi}_1^{(b)}(x) \psi_1^{(b)}(x)$ contracts with $\bar{\psi}_2^{(b)}(y) \psi_2^{(b)}(y)$ this amounts to a factor $\exp(-2(\chi, \delta_n(x) - \delta_n(y)))$ as can be seen from (5.7). Thus the factorization

$$C_1(\tau) = I(\tau) C_1(\tau)_{free} \quad , \quad (5.12)$$

holds. $C_1(\tau)_{free}$ simply denotes the replacement $\langle \dots \rangle_0 \rightarrow \langle \dots \rangle_{free}$ where the latter means expectation value with respect to free, massless fermions.

The factor $I(\tau)$ is the integral over gauge and auxiliary fields

$$I(\tau) = \int d\mu_{\tilde{Q}}[\varphi] d\mu_{\tilde{C}}[\theta]$$

$$\begin{aligned}
& \times \exp \left(-2 \sum_{b=1}^N \left[\sum_{i=1}^{m_b} (\chi, \delta_n(x_i^{(b)} + \hat{\tau}) + \sum_{i=1}^{m'_b} (\chi, \delta_n(x'_i{}^{(b)})) \right] \right) \\
& \times \exp \left(+2 \sum_{b=1}^N \left[\sum_{i=1}^{n_b} (\chi, \delta_n(y_i^{(b)} + \hat{\tau}) + \sum_{i=1}^{n'_b} (\chi, \delta_n(y'_i{}^{(b)})) \right] \right). \quad (5.13)
\end{aligned}$$

To keep my notation simple I introduce two sets of space-time variables $\{w_j\}$ and $\{z_j\}$ which are given by

$$\begin{aligned}
\{w_j\}_{j=1}^M &:= \{x_l^{(b)} + \hat{\tau}, x'_k{}^{(b)} \mid l = 1, \dots, n_b; k = 1, \dots, n'_b; b = 1, \dots, N\}, \\
\{z_j\}_{j=1}^M &:= \{y_l^{(b)} + \hat{\tau}, y'_k{}^{(b)} \mid l = 1, \dots, m_b; k = 1, \dots, m'_b; b = 1, \dots, N\}. \quad (5.14)
\end{aligned}$$

Due to (5.10) both sets contain the same number M

$$M := \sum_{b=1}^N (n_b + n'_b) = \sum_{b=1}^N (m_b + m'_b), \quad (5.15)$$

of elements. Using this notation and (5.13)

$$\begin{aligned}
I(\tau) &= \int d\mu_{\tilde{Q}}[\varphi] e^{-2 \frac{e\pi}{\pi+gN} \sum_{j=1}^M (\varphi, \delta_n(w_j) - \delta_n(z_j))} \times \int d\mu_{\tilde{C}}[\theta] e^{-2\sqrt{g} \sum_{j=1}^M (\theta, \delta_n(w_j) - \delta_n(z_j))} \\
&= \exp \left(2 \sum_{i,j=1}^M \left(\delta_n(w_i) - \delta_n(z_i), K [\delta_n(w_j) - \delta_n(z_j)] \right) \right), \quad (5.16)
\end{aligned}$$

where K denotes

$$K := \frac{e^2 \pi^2}{(\pi + gN)^2} \tilde{Q} + g \tilde{C}. \quad (5.17)$$

The functional integral was solved using (A.22) from Appendix A.2. Inserting the choice (4.57) for the δ -sequence, $I(\tau)$ becomes

$$\begin{aligned}
I(\tau) &= \exp \left(\int \frac{d^2 p}{(2\pi)^2} e^{-2 \frac{|p|}{n}} \hat{K}(p) 2 \sum_{i,j=1}^M \left(e^{-ipw_i} - e^{-ipz_i} \right) \left(e^{+ipw_j} - e^{+ipz_j} \right) \right) \\
&= \exp \left(\sum_{i,j=1}^M V(w_i - z_j) - \frac{1}{2} \sum_{i \neq j}^M V(w_i - w_j) - \frac{1}{2} \sum_{i \neq j}^M V(z_i - z_j) \right), \quad (5.18)
\end{aligned}$$

where the potential $V(x)$ is defined as (compare (4.54), (4.55) for the covariances \tilde{Q} and \tilde{C})

$$\begin{aligned}
V(x) &:= 2 \int \frac{d^2 p}{(2\pi)^2} e^{-2 \frac{|p|}{n}} \hat{K}(p) (2 - 2 \cos(px)) \\
&= 4 \int \frac{d^2 p}{(2\pi)^2} e^{-2 \frac{|p|}{n}} \frac{e^2 \pi^2}{(\pi + gN)^2} \frac{1}{p^2 + \frac{e^2 N}{\pi + gN}} \frac{1}{p^2} (1 - \cos(px))
\end{aligned}$$

$$+ 4 \int \frac{d^2 p}{(2\pi)^2} e^{-2\frac{|p|}{n}} g \frac{\pi}{\pi + gN} \frac{1}{p^2} (1 - \cos(px)) . \quad (5.19)$$

In both integrals the infrared problem is cured by the $(1 - \cos(px))$ term. The first one even has no ultraviolet problem, and it can be solved after the limit $n \rightarrow \infty$ was taken. The other one has to be evaluated for finite n . Using Appendix B.2 one obtains

$$\begin{aligned} V(x) &= \frac{2}{\pi} \frac{e^2 \pi^2}{(\pi + gN)^2} \frac{\pi + gN}{e^2 N} \left(\ln |x| + K_0 \left(\sqrt{\frac{e^2 N}{\pi + gN}} |x| \right) + \ln \left(\frac{1}{2} \sqrt{\frac{e^2 N}{\pi + gN}} \right) + \gamma \right) \\ &\quad + \frac{2}{\pi} g \frac{\pi}{\pi + gN} \left(\ln |x| + \ln \left(\frac{n}{4} \right) + O\left(\frac{1}{n}\right) \right) \\ &= \frac{1}{N} \ln |x| + \tilde{V}(x) + \frac{2\pi g}{\pi + gN} \ln \left(\frac{n}{4} \right) + O\left(\frac{1}{n}\right) , \end{aligned} \quad (5.20)$$

where I defined

$$\tilde{V}(x) := \frac{2\pi}{N(\pi + gN)} \left(K_0 \left(\sqrt{\frac{e^2 N}{\pi + gN}} |x| \right) + \ln \left(\frac{1}{2} \sqrt{\frac{e^2 N}{\pi + gN}} \right) + \gamma \right) . \quad (5.21)$$

Thus one ends up with

$$\begin{aligned} I(\tau) &= \left(\frac{n}{4} \right)^{\frac{\pi g}{\pi + gN} 2M} e^{O(\frac{1}{n})} \\ &\times \exp \left(\sum_{i,j=1}^M \tilde{V}(w_i - z_j) - \frac{1}{2} \sum_{i \neq j}^M \tilde{V}(w_i - w_j) - \frac{1}{2} \sum_{i \neq j}^M \tilde{V}(z_i - z_j) \right) \\ &\times \prod_{i,j=1}^M (w_i - z_j)^2 \prod_{i < j}^M (w_i - w_j)^{-2} (z_i - z_j)^{-2} . \end{aligned} \quad (5.22)$$

As announced, the n dependent factor can be absorbed in a wave function renormalization constant Z for the chiral densities $\bar{\psi}^{(a)} P_{\pm} \psi^{(a)}$

$$Z := \left(\frac{4}{n} \right)^{\frac{\pi g}{\pi + gN}} . \quad (5.23)$$

(5.22) allows a discussion of the large τ behaviour of $I(\tau)$. $\tilde{V}(x)$ depends on x only via the modified Bessel function K_0 . Since K_0 approaches zero exponentially, $\exp(\tilde{V}(x))$ goes to a constant for large τ , and the only remaining τ dependence for large τ of $I(\tau)$ must come from the rational function of the space-time arguments. Inserting the sets $\{w_j\}, \{z_j\}$ (see (5.14)) and extracting the leading power in τ gives

$$I(\tau) = F(\{w_j\}, \{z_j\}) \times (\tau^2)^{-\frac{1}{N} \sum_{b,b'=1}^N (n_b - m_b)(n'_{b'} - m'_{b'})} \left(1 + O\left(\frac{1}{\tau}\right) \right) , \quad (5.24)$$

where $F(\{w_j\}, \{z_j\})$ is some function depending on the space time arguments within the clusters.

$C_1(\tau)_{free}$ factorizes with respect to the flavors

$$C_1(\tau)_{free} = s \prod_{b=1}^N \left\langle \prod_{i=1}^{n_b} \psi_1^{(b)}(x_i^{(b)} + \hat{\tau}) \prod_{i=1}^{m'_b} \bar{\psi}_2^{(b)}(y_i'^{(b)}) \prod_{i=1}^{n'_b} \psi_1^{(b)}(x_i'^{(b)}) \prod_{i=1}^{m_b} \bar{\psi}_2^{(b)}(y_i^{(b)} + \hat{\tau}) \right. \\ \left. \times \prod_{i=1}^{m'_b} \psi_2^{(b)}(y_i'^{(b)}) \prod_{i=1}^{n_b} \bar{\psi}_1^{(b)}(x_i^{(b)} + \hat{\tau}) \prod_{i=1}^{m_b} \psi_2^{(b)}(y_i^{(b)} + \hat{\tau}) \prod_{i=1}^{n'_b} \bar{\psi}_1^{(b)}(x_i'^{(b)}) \right\rangle_{free} . \quad (5.25)$$

Using the explicit form of the free propagator G^o (Appendix A.1) it can be expressed in terms of determinants. The general structure of a factor with fixed flavor (flavor indices suppressed) is

$$\left\langle \prod_{i=1}^n \psi_1(w_i) \bar{\psi}_2(z_i) \prod_{i=1}^n \psi_2(z_i) \bar{\psi}_1(w_i) \right\rangle_{free} \\ = \sum_{\pi(n)} \text{sign}(\pi) G_{12}^o(w_1 - z_{\pi(1)}) G_{12}^o(w_2 - z_{\pi(2)}) \dots G_{12}^o(w_n - z_{\pi(n)}) \\ \times \sum_{\pi(n)} \text{sign}(\pi) G_{21}^o(z_1 - w_{\pi(1)}) G_{21}^o(z_2 - w_{\pi(2)}) \dots G_{21}^o(z_n - w_{\pi(n)}) \\ = (-1)^n \left(\frac{1}{2\pi} \right)^{2n} \left| \sum_{\pi(n)} \text{sign}(\pi) \frac{1}{\tilde{w}_1 - \tilde{z}_{\pi(1)}} \dots \frac{1}{\tilde{w}_n - \tilde{z}_{\pi(n)}} \right|^2 \\ = \left(\frac{1}{2\pi} \right)^{2n} \left| \det_{(i,j)} \left(\frac{1}{\tilde{w}_i - \tilde{z}_j} \right) \right|^2 . \quad (5.26)$$

Determinants of this type can be rewritten using Cauchy's identity (see e.g. [22])

$$\det_{(i,j)} \left(\frac{1}{\tilde{w}_i - \tilde{z}_j} \right) = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{1 \leq i < j \leq n} (\tilde{w}_i - \tilde{w}_j)(\tilde{z}_i - \tilde{z}_j)}{\prod_{i,j=1}^n (\tilde{w}_i - \tilde{z}_j)} . \quad (5.27)$$

Hence one obtains

$$C_1(\tau)_{free} = \\ s' \left(\frac{1}{2\pi} \right)^{2M} \prod_{b=1}^N \prod_{i,j=1}^{n_b+n'_b} \left(w_i^{(b)} - z_j^{(b)} \right)^{-2} \prod_{1 \leq i < j \leq n_b+n'_b} \left(w_i^{(b)} - w_j^{(b)} \right)^2 \left(z_i^{(b)} - z_j^{(b)} \right)^2 , \quad (5.28)$$

where the sets $\{w_j^{(b)}\}, \{z_j^{(b)}\}$ for fixed flavor b are given by

$$\{w_j^{(b)}\}_{j=1}^{n_b+n'_b} := \{x_l^{(b)} + \hat{\tau}, x_k'^{(b)} \mid l = 1, \dots, n_b; k = 1, \dots, n'_b\} ,$$

$$\{z_j^{(b)}\}_{j=1}^{m_b+m'_b} := \{y_l^{(b)} + \hat{\tau}, y'_k{}^{(b)} \mid l = 1, \dots, m_b; k = 1, \dots, m'_b\} . \quad (5.29)$$

Note that $m_b + m'_b = n_b + n'_b$ due to (5.10). Inserting the sets (5.29), one can extract the large τ behaviour

$$C_1(\tau)_{free} = s' \left(\frac{1}{2\pi} \right)^{2M} (\tau^2)^{\sum_{b=1}^N (n_b - m_b)(n'_b - m'_b)} \left(1 + O\left(\frac{1}{\tau}\right) \right) . \quad (5.30)$$

Thus (use (5.12), (5.24) and (5.30))

$$C_1(\tau) \propto \left(\frac{1}{\tau^2} \right)^E \left(1 + O\left(\frac{1}{\tau}\right) \right) , \quad (5.31)$$

where the exponent E is given by

$$E := \frac{1}{N} \sum_{b,b'=1}^N (n_b - m_b)(n'_{b'} - m'_{b'}) - \sum_{b=1}^N (n_b - m_b)(n'_b - m'_b) . \quad (5.32)$$

Using (5.10) one can rewrite E

$$\begin{aligned} E &= \sum_{b=1}^N (n_b - m_b)(n_b - m_b) - \frac{1}{N} \sum_{b,b'=1}^N (n_b - m_b)(n_{b'} - m_{b'}) = \\ &= \frac{1}{N} \sum_{b,b'=1}^N (n_b - m_b) R_{bb'} (n_{b'} - m_{b'}) . \end{aligned} \quad (5.33)$$

The matrix R

$$R_{bb'} =: \delta_{bb'} N - 1 , \quad (5.34)$$

is discussed in Appendix B.3. There the corresponding eigenvalue problem is solved. One finds one eigenvalue 0, and $N - 1$ eigenvalues N . The eigenvector x^0 to the eigenvalue 0 is given by $x^0 = 1/\sqrt{N}(1, 1, \dots, 1)^T$. Hence the quadratic form $x^T R x$ is positive semidefinite, and vanishes only if x is a multiple of x^0 . This implies that the exponent E is nonnegative and vanishes only for

$$n_b - m_b = m'_b - n'_b = n \quad \forall b = 1, 2, \dots, N, \quad n \in \mathbb{Z} . \quad (5.35)$$

All those possibilities lead to a nonvanishing limit $C_1(\infty) := \lim_{\tau \rightarrow \infty} C_1(\tau)$. In some of the cases $C_1(\infty)$ will be cancelled by C_2 which is given by

$$\begin{aligned} C_2 &= \left\langle \prod_{b=1}^N \prod_{i=1}^{n_b} \bar{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{i=1}^{m_b} \bar{\psi}^{(b)}(y_i^{(b)}) P_- \psi^{(b)}(y_i^{(b)}) \right\rangle_0 \\ &\times \left\langle \prod_{b=1}^N \prod_{i=1}^{n'_b} \bar{\psi}^{(b)}(x'_i{}^{(b)}) P_+ \psi^{(b)}(x'_i{}^{(b)}) \prod_{i=1}^{m'_b} \bar{\psi}^{(b)}(y'_i{}^{(b)}) P_- \psi^{(b)}(y'_i{}^{(b)}) \right\rangle_0 , \end{aligned} \quad (5.36)$$

Using (5.7), (5.9) it is possible to find a necessary condition similar to (5.10) that has to be fulfilled in order to be able to contract the fermions entirely. C_2 does not vanish only for

$$n_b = m_b \quad \text{and} \quad n'_b = m'_b, \quad b = 1, \dots, N. \quad (5.37)$$

In these cases C_2 then cancels $C_1(\infty)$ and the operators cluster. Thus violation of clustering of C_τ is expressed in the condition

$$n_b - m_b = m'_b - n'_b = n, \quad \forall b = 1, 2, \dots, N, \quad n \in \mathbb{Z} \setminus \{0\}. \quad (5.38)$$

How does this picture change when one allows vector currents as well? First I notice that vector currents do not contribute to the φ and θ integrals. Consider e.g. the term $\bar{\psi}_1^{(1)}(x)\psi_2^{(2)}(x)$ showing up in a current that mixes flavors 1 and 2². The $\bar{\psi}_1^{(1)}(x)$ enters a propagator $G_{21}(\cdot, x)$, $\psi_2^{(2)}(x)$ a propagator $G_{21}(x, \cdot)$. Inspecting (5.7) immediately shows the cancellation of the χ dependence in the product of the propagators. Hence each vector current can only contribute a $1/\tau$ from the free propagator.

Nonvanishing results remain only if the flavors that occur in the vector currents can contract entirely. Thus one has to consider only ‘closed cycles’ like e.g.

$$\bar{\psi}^{(1)}(x)\gamma_\mu\psi^{(2)}(x) \quad \bar{\psi}^{(2)}(y)\gamma_\nu\psi^{(3)}(y) \quad \bar{\psi}^{(3)}(z)\gamma_\omega\psi^{(1)}(z). \quad (5.39)$$

In principle there are two possibilities to distribute the space-time arguments x, y, z . If they are all in one cluster they do not bring in any τ -dependence. They do not modify the clustering, only the constant C . If one distributes the closed cycle over both clusters then the situation changes. Each vector current with a partner in the other cluster contributes a factor $1/\tau$ from the free propagator. This implies that any combination of vector currents alone clusters. Nevertheless a combination of vector currents together with the ansatz (5.3) could violate clustering. But the gauge field integral contributions from the chiral charges $\bar{\psi}P_\pm\psi$ can at most compensate the $1/\tau$ of these charges, nothing else ($x^T R x$ is positive semidefinite, compare (5.34) and the Appendix B.3). Hence adding vector currents that can only contract between the clusters can at most enforce operators to cluster, never create extra powers of τ that lead to violation of clustering. The same is true when inserting currents containing only a single flavor where one has to introduce a point splitting regulator. The gauge field transporter that connects $\bar{\psi}^{(b)}(x - \varepsilon)$ and $\psi^{(b)}(x + \varepsilon)$ is only a modification within a cluster that does not change the clustering behaviour.

²Together with $\bar{\psi}_2^{(1)}(x)\psi_1^{(2)}(x)$ this is generic due to the off diagonal choice of the γ -algebra (see Appendix B.1).

5.3 Symmetry properties of operators that violate clustering

In this section the symmetry properties of the nonclustering operators will be analyzed. To make the notation more convenient I introduce

$$\mathcal{O}_{\pm}(\{x\}) := \prod_{b=1}^N \bar{\psi}^{(b)}(x^{(b)}) P_{\pm} \psi^{(b)}(x^{(b)}) . \quad (5.40)$$

It will turn out that the lack of clustering of \mathcal{O}_{\pm} is related to the fact that they are singlets³ under the conserved symmetry group $U(1)_V \times SU(N)_L \times SU(N)_R$, but transform nontrivially under the explicitly broken $U(1)_A$. To obtain operators that transform under a single irreducible representation of the symmetry group namely the trivial representation, I antisymmetrize \mathcal{O}_{\pm} with respect to the flavor indices and call the result \mathcal{O}_{\pm}^a .

$$\begin{aligned} \mathcal{O}_{\pm}^a(\{x\}) := & (-1)^{\frac{N(N-1)}{2}} \left[\frac{1}{N!} \sum_{\pi} \text{sign}(\pi) \prod_{b=1}^N \bar{\psi}_{\frac{1}{2}}^{(\pi(b))}(x^{(b)}) \right] \\ & \left[\frac{1}{N!} \sum_{\pi'} \text{sign}(\pi') \prod_{b'=1}^N \psi_{\frac{1}{2}}^{(\pi'(b'))}(x^{(b')}) \right] . \end{aligned} \quad (5.41)$$

The global sign comes from shifting all $\bar{\psi}$ to the left. Using

$$\prod_{b=1}^N \psi_{\alpha}^{(b)}(x^{(\pi(b))}) = \text{sign}(\pi^{-1}) \prod_{b=1}^N \psi_{\alpha}^{(\pi^{-1}(b))}(x^{(b)}) , \quad (5.42)$$

one can express \mathcal{O}_{\pm}^a in terms of symmetrized space-time arguments as well,

$$\mathcal{O}_{\pm}^a(\{x\}) := \frac{1}{(N!)^2} \sum_{\pi, \pi'} \prod_{b=1}^N \bar{\psi}^{(b)}(x^{(\pi(b))}) P_{\pm} \psi^{(b)}(x^{(\pi'(b))}) . \quad (5.43)$$

Since the constant $C = \lim_{\tau \rightarrow \infty} C(\tau)$ is invariant under the permutation of arguments within a cluster (compare (5.22), (5.28)), the latter expression shows explicitly that the constant C is the same for \mathcal{O}_{\pm} and \mathcal{O}_{\pm}^a . From (5.41) one easily reads off the invariance of \mathcal{O}_{\pm}^a under

$$U(1)_V \times SU(N)_L \times SU(N)_R , \quad (5.44)$$

and nontrivial transformation properties under $U(1)_A$. I end up with the following picture of the clustering behaviour: The prototype of a correlation function that violates clustering is given by

$$C(\tau) = \left\langle \prod_{i=1}^n \mathcal{O}_{+}^a(\{x_i + \hat{\tau}\}) \prod_{i=1}^m \mathcal{O}_{-}^a(\{y_i + \hat{\tau}\}) \prod_{i=1}^{n'} \mathcal{O}_{+}^a(\{x'_i\}) \prod_{i=1}^{m'} \mathcal{O}_{-}^a(\{y'_i\}) \right\rangle$$

³ The condition that operators that violate clustering should be singlets under $SU(N)_L \times SU(N)_R$ was already discussed in [10].

$$- \left\langle \prod_{i=1}^n \mathcal{O}_+^a(\{x_i\}) \prod_{i=1}^m \mathcal{O}_-^a(\{y_i\}) \right\rangle \left\langle \prod_{i=1}^{n'} \mathcal{O}_+^a(\{x'_i\}) \prod_{i=1}^{m'} \mathcal{O}_-^a(\{y'_i\}) \right\rangle, \quad (5.45)$$

with the condition

$$n - m = -n' + m' \in \mathbb{Z} \setminus \{0\}. \quad (5.46)$$

Insertion of closed cycles of vector currents into a cluster does not change the clustering behaviour. If one inserts vector currents that can contract only to a partner in the other cluster, operators that violated clustering before now become operators obeying the cluster property. Of course it is possible to generalize the operators \mathcal{O}_\pm further by e.g. splitting the arguments and connect them with a parallel transporter. Since this is a modification within a cluster, the extra terms in the functional integral will not depend on τ and only modify the constant.

For completeness I determine the constant $C = \lim_{\tau \rightarrow \infty} C(\tau)$ with $C(\tau)$ defined in (5.45)

$$C = \mathcal{F}(\{x\}, \{y\}) \mathcal{F}(\{x'\}, \{y'\}). \quad (5.47)$$

In the limit $\tau \rightarrow \infty$ the dependence on the space time arguments factorizes into two parts that depend on the arguments in the two clusters. This function \mathcal{F} is unambiguous for an operator, if only the other operator has the right quantum numbers to form a pair that violates clustering, otherwise it vanishes. It is given by

$$\begin{aligned} \mathcal{F}(\{x\}, \{y\}) &= \prod_{b=1}^N \frac{\prod_{1 \leq i < j \leq n} (x_i^{(b)} - x_j^{(b)})^2 \prod_{1 \leq i < j \leq m} (y_i^{(b)} - y_j^{(b)})^2}{\prod_{i=1}^n \prod_{j=1}^m (x_i^{(b)} - y_j^{(b)})} \\ &\times e^{\sum \tilde{V}(x_i^{(b)} - y_j^{(b')}) - \frac{1}{2} \sum (1 - \delta_{bb'} \delta_{ij}) (\tilde{V}(x_i^{(b)} - x_j^{(b')}) + \tilde{V}(y_i^{(b)} - y_j^{(b')}))} \\ &\times \left(\frac{1}{2\pi} \right)^{N(n+m)} \left[\frac{e^2 N}{4(\pi + gN)} e^{2\gamma} \right]^{\frac{\pi}{\pi + gN} \frac{N(n-m)^2}{2}}. \end{aligned} \quad (5.48)$$

\sum denotes summation over all possible values of the indices b, b', i, j within the clusters. \tilde{V} is defined as (compare (5.20))

$$\tilde{V}(x) := \frac{1}{N} \ln(x^2) + \tilde{V}(x). \quad (5.49)$$

5.4 Decomposition into clustering states

In the last section it was shown that operators that violate clustering are singlets under $U(1)_V \times SU(N)_L \times SU(N)_R$, but transform nontrivially under

$U(1)_A$. Thus the decomposition of the vacuum state in terms of clustering θ -vacua⁴ can make use of the charge that is associated to the axial transformation $U(1)_A$

$$\psi^{(b)} \longrightarrow e^{i\varepsilon\gamma_5}\psi^{(b)} . \quad (5.50)$$

An arbitrary product \mathcal{B} of $\bar{\psi}_\alpha^{(b)}(x), \psi_{\alpha'}^{(b')}(x')$ transforms under $U(1)_A$ as

$$\mathcal{B}(\{x\}) \longrightarrow e^{im\varepsilon}\mathcal{B}(\{x\}) , \quad m \in \mathbb{Z} . \quad (5.51)$$

(More generally one can consider observables that are sums of operators with definite transformation properties under $U(1)_A$). Define the corresponding charge $Q_5(\mathcal{B})$ as m . Obviously \mathcal{O}_\pm^a (see (5.41)) have the charge $\pm 2N$. I will now decompose the expectation functional $\langle \cdot \rangle_0$ into states $\langle \cdot \rangle_0^\theta$ labeled by a parameter $\theta \in [-\pi, \pi]$ defined as follows

$$\langle \mathcal{B}(\{x\}) \rangle_0^\theta := e^{i\theta \frac{Q_5(\mathcal{B})}{2N}} \lim_{\tau \rightarrow \infty} \langle \mathcal{U}_\tau(\mathcal{B}) \mathcal{B}(\{x\}) \rangle_0 . \quad (5.52)$$

The set of ‘test operators’ $\mathcal{U}_\tau(\mathcal{B})$ is defined by

$$\mathcal{U}_\tau(\mathcal{B}) := \begin{cases} \mathcal{N}^{(n)}(\{y\}) \prod_{i=1}^n \mathcal{O}_\mp^a(\{y + \hat{\tau}\}) \text{ for } Q_5(\mathcal{B}) = \pm 2nN, \quad n \geq 1 , \\ 1 \text{ otherwise} . \end{cases} \quad (5.53)$$

Up to the requirement of being nondegenerate, the arguments $\{y\}$ are arbitrary. The normalizing factor $\mathcal{N}^{(n)}(\{y\})$ is defined such that

$$\lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{B}^\dagger) \mathcal{U}_\tau(\mathcal{B}) \rangle_0 = 1 . \quad (5.54)$$

It can be read off from (5.48)

$$\begin{aligned} \mathcal{N}^{(n)}(\{y\}) &= \left(\frac{1}{2\pi} \right)^{-Nn} \left[\frac{e^2 N}{4(\pi + gN)} e^{2\gamma} \right]^{-\frac{\pi}{\pi + gN} \frac{Nn^2}{2}} \\ &\times \prod_{b=1}^N \prod_{1 \leq i < j \leq n} \left(y_i^{(b)} - y_j^{(b)} \right)^{-2} e^{\frac{1}{2} \sum_{b,b',i,j} (1 - \delta_{bb'} \delta_{ij}) \tilde{V} \left(y_i^{(b)} - y_j^{(b')} \right)} . \end{aligned} \quad (5.55)$$

The expectation functionals (states) $\langle \cdot \rangle_0^\theta$ have the following properties:

Theorem 5.1 :

i) The state $\langle \cdot \rangle_0$ constructed initially is recovered by averaging over θ

$$\langle \cdot \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \cdot \rangle_0^\theta d\theta . \quad (5.56)$$

⁴The term θ -vacua will be used here as well, but it has to be remarked that the construction here is completely different from the concept discussed in Section 2.1.

ii) The cluster decomposition property holds.

Proof:

i): The averaging procedure leaves a nonvanishing result only for operators \mathcal{B} with vanishing charge $Q_5(\mathcal{B}) = 0$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \mathcal{B} \rangle_0^\theta d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta \frac{Q_5(\mathcal{B})}{2N}} d\theta \lim_{\tau \rightarrow \infty} \langle \mathcal{U}_\tau(\mathcal{B}) \mathcal{B} \rangle_0 = \delta_{Q_5(\mathcal{B}),0} \langle \mathcal{B} \rangle = \langle \mathcal{B} \rangle_0 . \quad (5.57)$$

In the second step I used $\mathcal{U}_\tau(\mathcal{B}) = 1$ if $Q_5(\mathcal{B}) = 0$. To complete the proof, one has to check that in the state $\langle \cdot \rangle_0$ constructed initially the vacuum expectation values of operators \mathcal{B} with $Q_5(\mathcal{B}) \neq 0$ vanish (cf. also [30]). In the chosen representation of the γ -matrices this can be seen immediately. $Q_5(\mathcal{B}) \neq 0$ means that the number of $\bar{\psi}_1, \psi_1$ is not equal to the number $\bar{\psi}_2, \psi_2$. Since $G_{\alpha\alpha} = 0$, for each ψ_1 there has to be a $\bar{\psi}_2$ and for each ψ_2 a $\bar{\psi}_1$ to give a nonvanishing contribution. But this implies that the number of fields $\bar{\psi}_1, \psi_1$ in \mathcal{B} is equal to the number of fields $\bar{\psi}_2, \psi_2$. Hence

$$\langle \mathcal{B} \rangle_0 = 0 \text{ for } Q_5(\mathcal{B}) \neq 0 , \quad (5.58)$$

and the last equality in (5.57) holds.

ii): Let \mathcal{A} and \mathcal{B} be arbitrary operators. Define

$$C_\theta(\tau) := \lim_{\tau \rightarrow \infty} \left[\langle \mathcal{A}(\tau) \mathcal{B}(0) \rangle_0^\theta - \langle \mathcal{A}(0) \rangle_0^\theta \langle \mathcal{B}(0) \rangle_0^\theta \right] . \quad (5.59)$$

The dependence on the space-time arguments $\{x\}$ is not displayed explicitly, only the dependence on the shift variable τ . Depending on the axial charge Q_5 of the operators $\mathcal{A}, \mathcal{B}, \mathcal{AB}$, one has to insert the different alternatives for $\langle \cdot \rangle_0^\theta$ given in (5.53). I introduce the following convenient notation: An operator for which the first alternative in (5.53) holds I call a ‘type **I**’ operator, the operators where the second alternative holds is called ‘type **II**’. It has to be remarked that all operators of type **II** cannot have the problem of violating the clustering condition. Even among the type **I** operators there are examples that are not able to violate clustering (like e.g. $(\bar{\psi}^{(b)} P_+ \psi^{(b)})^N$, for fixed b , which does not contain a $SU(N)_L \times SU(N)_R$ singlet part). The operators with the structure $\prod^n \mathcal{O}_+ \prod^m \mathcal{O}_- \prod J_\mu^{(l)}$, $n - m \neq 0$ I call ‘type **V**’ for violating.

The following cases have to be distinguished

case #	$Q_5(\mathcal{A})$	$Q_5(\mathcal{B})$	$Q_5(\mathcal{AB})$
1	II	II	II
2	II ($\neq 0$)	II ($\neq 0$)	I
3	II ($\neq 0$)	I	II
4	II ($= 0$)	I	I
5	I ($= q$)	I ($= -q$)	II ($= 0$)
6	I	I	I

In parenthesis (\cdot) I denoted facts that necessarily follow. For example in Case 2 the requirement $Q_5(\mathcal{A}) \in \mathbf{II}, Q_5(\mathcal{B}) \in \mathbf{II}, Q_5(\mathcal{AB}) \in \mathbf{I}$ implies $Q_5(\mathcal{A}) \neq 0$ and $Q_5(\mathcal{B}) \neq 0$. If one of these two had charge zero, the other operator would be of type \mathbf{I} , since $Q_5(\mathcal{AB}) = Q_5(\mathcal{A}) + Q_5(\mathcal{B})$.

Case 1:

$$C_\theta(\tau) = \langle \mathcal{A}(\tau)\mathcal{B} \rangle_0 - \langle \mathcal{A} \rangle_0 \langle \mathcal{B} \rangle_0 \xrightarrow{\tau \rightarrow \infty} 0, \quad (5.60)$$

since if \mathcal{A} and \mathcal{B} are of type \mathbf{II} , they do not form a pair that violates clustering.

Case 2:

$$\begin{aligned} C_\theta(\tau) &= \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{AB})\mathcal{A}(\tau)\mathcal{B} \rangle_0 - \langle \mathcal{A} \rangle_0 \langle \mathcal{B} \rangle_0 \xrightarrow{\tau \rightarrow \infty} \\ &\langle \mathcal{A} \rangle_0 \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{AB})\mathcal{B} \rangle_0 - \langle \mathcal{A} \rangle_0 \langle \mathcal{B} \rangle_0 = 0, \end{aligned} \quad (5.61)$$

where I used the fact that $\langle \mathcal{A} \rangle_0$ factorizes, since \mathcal{A} is type \mathbf{II} and $\langle \mathcal{A} \rangle_0 = 0$ since $Q_5(\mathcal{A}) \neq 0$ (compare the note in the table). The interchange of the τ, τ' limits is justified, since all the functions involved are continuous in these variables and bounded (the exponent E in Equation (5.31) is nonnegative).

Case 3:

$$C_\theta(\tau) = \langle \mathcal{A}(\tau)\mathcal{B} \rangle_0 - \langle \mathcal{A} \rangle_0 \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{B})\mathcal{B} \rangle_0 \xrightarrow{\tau \rightarrow \infty} 0, \quad (5.62)$$

for the same reasons as in the last case.

Case 4:

$$\begin{aligned} C_\theta(\tau) &= \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{AB})\mathcal{A}(\tau)\mathcal{B} \rangle_0 - \langle \mathcal{A} \rangle_0 \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{B})\mathcal{B} \rangle_0 \xrightarrow{\tau \rightarrow \infty} \\ &\langle \mathcal{A} \rangle_0 \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{AB})\mathcal{B} \rangle_0 - \langle \mathcal{A} \rangle_0 \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{B})\mathcal{B} \rangle_0 = 0. \end{aligned} \quad (5.63)$$

Again $\langle \mathcal{A} \rangle_0$ factorizes, and $\mathcal{U}_{\tau'}(\mathcal{AB}) = \mathcal{U}_{\tau'}(\mathcal{B})$ since $Q_5(\mathcal{AB}) = Q_5(\mathcal{B})$.

Case 5:

$$\begin{aligned} C_\theta(\tau) &= \langle \mathcal{A}(\tau)\mathcal{B} \rangle_0 - \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{A})\mathcal{A} \rangle_0 \lim_{\tau'' \rightarrow \infty} \langle \mathcal{U}_{\tau''}(\mathcal{B})\mathcal{B} \rangle_0 \xrightarrow{\tau \rightarrow \infty} \\ &\begin{cases} \mathcal{F}_\mathcal{A}\mathcal{F}_\mathcal{B} - \mathcal{F}_\mathcal{A}\mathcal{F}_\mathcal{B} = 0 & \text{for } \mathcal{A}, \mathcal{B} \text{ of V-type,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (5.64)$$

In the first case I used the factorization of the argument function $\mathcal{F}_{\mathcal{AB}}$ introduced in (5.48).

Case 6:

$$\begin{aligned} C_\theta(\tau) &= \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{AB})\mathcal{A}(\tau)\mathcal{B} \rangle_0 - \lim_{\tau' \rightarrow \infty} \langle \mathcal{U}_{\tau'}(\mathcal{A})\mathcal{A} \rangle_0 \lim_{\tau'' \rightarrow \infty} \langle \mathcal{U}_{\tau''}(\mathcal{B})\mathcal{B} \rangle_0 \xrightarrow{\tau \rightarrow \infty} \\ &\begin{cases} \mathcal{F}_\mathcal{A} \lim_{\tau' \rightarrow \infty} \mathcal{F}_{\mathcal{U}_{\tau'}\mathcal{B}} - \mathcal{F}_\mathcal{A}\mathcal{F}_\mathcal{B} = 0 & \text{for } \mathcal{A}, \mathcal{B} \text{ of V-type,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (5.65)$$

To justify the first case, one has to show that $\lim_{\tau' \rightarrow \infty} \mathcal{F}_{\mathcal{U}_{\tau'}\mathcal{B}} = \mathcal{F}_\mathcal{B}$. This can be seen immediately from (5.48). Shifting τ' in $\mathcal{F}_{\mathcal{U}_{\tau'}\mathcal{B}}$ corresponds to shifting

the arguments of the test operator $\mathcal{U}_{\tau'}(\mathcal{B})$. For example this could be the set (referring to (5.48) for \mathcal{F})

$$\left\{ x_i^{(b)} \mid n' < i \leq n; b = 1, 2, \dots, N \right\} , \quad (5.66)$$

where $n' < n$ depends on the charge $Q_5(\mathcal{B})$. The τ' terms cancel for $\tau' \rightarrow \infty$, and what remains is $\mathcal{F}_{\mathcal{B}}$, since the normalization of $\mathcal{U}_{\tau'}(\mathcal{B})$ cancels exactly the extra terms. \square

This concludes the analysis of the vacuum structure of the massless model. The decomposition performed here is equivalent to what is hoped to have been obtained by the superposition of topological sectors to the θ -vacuum. The procedure adopted here has the advantage of being in a better mathematical status. Finally it has to be remarked that it is in full agreement with the picture that is conventionally deduced from the discussions of topological sectors on a compact manifold [40], [51].

One now can compare properties of the vacuum state in QED_2 to the expected properties of the instanton construction of QCD. This opens a series of lessons for the topics discussed in Chapter 2 that I will draw from the model.

Lesson 1 :

The structure of the vacuum functional that has been suggested within the instanton picture is recovered.

In particular the formulas (2.41), (2.42) are confirmed. By looking at the prescription (5.53) one sees that only operators with chirality $2N\nu$, $\nu \in \mathbb{Z}$ have non vanishing vacuum expectation values, as has been claimed by 't Hooft for QCD. Also the phase of (2.41) comes out correctly, if θ is defined mod (2π) . It has to be stressed that the vacuum state in QED_2 has been constructed without making use of topologically nontrivial configurations but nevertheless has the properties expected from this picture.

Chapter 6

Bosonization and vector currents

In this chapter it will be shown that expectation values $\langle \dots \rangle_0^\theta$ of certain operators (vector currents of the Cartan subalgebra, chiral densities) can be rewritten in terms of expectation values of a bosonic theory. Furthermore I will comment on currents that are not of the Cartan type. I will derive two theorems on n-point functions of vector currents in the model with vanishing fermion masses. This will clarify the structure of the Hilbert space of the massless model.

6.1 Evaluation of a generalized generating functional

I evaluate the following generating functional

$$E(n_b, m_b; a^{(b)}) := \left\langle \prod_{b=1}^N \prod_{i=1}^{n_b} \bar{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{j=1}^{m_b} \bar{\psi}^{(b)}(y_j^{(b)}) P_- \psi^{(b)}(y_j^{(b)}) e^{ie \sum_{b=1}^N (a_\mu^{(b)} j_\mu^{(b)})} \right\rangle_0^\theta. \quad (6.1)$$

Obviously this is a simple modification of expectation values already considered in the last chapter. The insertion of the chiral densities $\bar{\psi}^{(b)} P_\pm \psi^{(b)}$ is motivated by the expansion (3.37) of the mass term of the action. Setting $n_b = m_b = 0$ for $b = 1, 2, \dots, N$ reduces $E(n_b, m_b; a^{(b)})$ to the generating functional for vector currents in the model with vanishing fermion masses.

To work out the dependence on the sources $a^{(b)}$ I first consider the case

$$n_b - m_b = 0 \quad b = 1, 2, \dots, N, \quad (6.2)$$

where the θ -prescription is remarkably simple, i.e. it coincides with the naive expectation value (compare (5.53)).

As in the last chapter the expectation value E factorizes

$$E(n_b, n_b; a^{(b)}) := I(n_b, n_b; a^{(b)}) \times E_{free}(n_b, n_b) , \quad (6.3)$$

where the result for the free expectation value E_{free} can immediately be taken over from (5.28)

$$\begin{aligned} E_{free}(n_b, n_b) &:= \left\langle \prod_{b=1}^N \prod_{i=1}^{n_b} \bar{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{j=1}^{n_b} \bar{\psi}^{(b)}(y_j^{(b)}) P_- \psi^{(b)}(y_j^{(b)}) \right\rangle_{free} = \\ &\left(\frac{1}{2\pi} \right)^{2 \sum_b n_b} \prod_{b=1}^N \prod_{i,j=1}^{n_b} \left(x_i^{(b)} - y_j^{(b)} \right)^{-2} \prod_{1 \leq i < j \leq n_b} \left(x_i^{(b)} - x_j^{(b)} \right)^2 \left(y_i^{(b)} - y_j^{(b)} \right)^2 . \end{aligned} \quad (6.4)$$

The second factor I from the functional integration over the fields A and h' reads (compare (5.13))

$$\begin{aligned} I(n_b, n_b; a^{(b)}) &:= \\ &\left\langle \exp \left(-2 \sum_{b=1}^N \sum_{j=1}^{n_b} \left(\chi^{(b)}, \delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right) \right) \exp \left(ie \sum_{b=1}^N (a_\mu^{(b)}, j_\mu^{(b)}) \right) \right\rangle_0 , \end{aligned} \quad (6.5)$$

where the second exponential collects the terms from the propagators, and $\chi^{(b)}$ is given by (see (4.52))

$$\chi^{(b)}(x) = \frac{e\pi}{\pi + gN} \varphi(x) + \sqrt{g} \theta(x) + e \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} a_\nu^{(b)}(x) . \quad (6.6)$$

As discussed in Section 3.4 the external sources $a_\mu^{(b)}$ can be treated by including them into the fermion determinant. One can take over the result (4.49) from Section 4.3 to obtain

$$\begin{aligned} I(n_b, n_b; a^{(b)}) &= \int d\mu_Q[A] d\mu_C[h'] \exp \left(-2 \sum_{b=1}^N \sum_{j=1}^{n_b} \left(\chi^{(b)}, \delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right) \right) \\ &\times \exp \left(-\frac{e^2}{\pi + gN} \left(A, T \sum_{b=1}^N a^{(b)} \right) - \frac{e\sqrt{g}}{\pi} \left(h', T \sum_{b=1}^N a^{(b)} \right) - \frac{e^2}{2\pi} \sum_{b=1}^N \left(a^{(b)}, T a^{(b)} \right) \right) , \end{aligned} \quad (6.7)$$

where the fields φ and A_μ and θ and h'_μ respectively are related by (c.f. (4.53))

$$\varphi(x) := \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} A_\nu(x) \quad , \quad \theta(x) := \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} h'_\nu(x) . \quad (6.8)$$

This can be used to rewrite everything in terms of the scalar fields φ and θ

$$\left(A_\mu, T a_\mu^{(b)} \right) = - \left(\varphi, \varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)} \right) \quad , \quad \left(h'_\mu, T a_\mu^{(b)} \right) = - \left(\theta, \varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)} \right) . \quad (6.9)$$

Thus one ends up with

$$I(n_b, n_b; a^{(b)}) = \int d\mu_{\tilde{Q}}[\varphi] d\mu_{\tilde{C}}[\theta] \exp \left(-2 \sum_{b=1}^N \sum_{j=1}^{n_b} \left(\chi^{(b)}, \delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right) \right) \times \\ \exp \left(\frac{e^2}{\pi + gN} \left(\varphi, T \sum_{b=1}^N \varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)} \right) + \frac{e\sqrt{g}}{\pi} \left(\theta, T \sum_{b=1}^N \varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)} \right) - \frac{e^2}{2\pi} \sum_{b=1}^N \left(a^{(b)}, T a^{(b)} \right) \right). \quad (6.10)$$

The Gaussian integrals over φ and θ can be solved and one obtains

$$I(n_b, n_b; a^{(b)}) = \exp \left(-\frac{e^2}{2\pi} \sum_{b=1}^N \left(a^{(b)}, T a^{(b)} \right) \right) \\ \times \exp \left(\frac{1}{2} \sum_{b,b'=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)}, \left[\left(\frac{e^2}{\pi + gN} \right)^2 \tilde{Q} + \frac{e^2 g}{\pi^2} \tilde{C} \right] \varepsilon_{\rho\sigma} \partial_\rho a_\sigma^{(b')} \right) \right) \\ \times \exp \left(-2e \sum_{b=1}^N \sum_{j=1}^{n_b} \left(\frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} a_\nu^{(b)}, \delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right) \right) \\ \times \exp \left(-2e \sum_{b,b'=1}^N \sum_{j=1}^{n_b} \left(\varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b')}, \left[\frac{e^2 \pi}{(\pi + gN)^2} \tilde{Q} + \frac{g}{\pi} \tilde{C} \right] \delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right) \right) \\ \times \exp \left(2 \sum_{b,b'=1}^N \sum_{j,j'=1}^{n_b, n_{b'}} \left(\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right), \left[\left(\frac{e\pi}{\pi + gN} \right)^2 \tilde{Q} + g \tilde{C} \right] \left[\delta_n(x_{j'}^{(b')}) - \delta_n(y_{j'}^{(b')}) \right] \right). \quad (6.11)$$

The first step in the analysis of this expression is to simplify the exponents quadratic in $a^{(b)}$. Using

$$\left(a_\mu^{(b)}, T_{\mu\nu} a_\nu^{(b)} \right) = \left(\varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)}, \frac{-1}{\Delta} \varepsilon_{\rho\sigma} \partial_\rho a_\sigma^{(b)} \right), \quad (6.12)$$

one can rewrite the quadratic form for $a_\mu^{(b)}$ in a quadratic form for $\varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)}$. The corresponding term of the exponent now reads

$$-\frac{e^2}{2\pi} \sum_{b,b'=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu a_\nu^{(b)}, M_{bb'} \varepsilon_{\rho\sigma} \partial_\rho a_\sigma^{(b')} \right), \quad (6.13)$$

where the covariance M is given by

$$M = \frac{1}{-\Delta + \frac{e^2 N}{\pi + gN}} \left[\frac{\pi}{\pi + gN} \mathbb{I} + \frac{g}{\pi + gN} R \right] + \frac{e^2}{\pi + gN} \frac{-1}{\Delta - \Delta + \frac{e^2 N}{\pi + gN}} R. \quad (6.14)$$

The numerical matrix

$$R_{bb'} = \delta_{b,b'} N - 1 \quad , \quad b, b' = 1, \dots, N, \quad (6.15)$$

is analyzed in Appendix B.3. There I quote explicitly the orthogonal matrix U that diagonalizes R

$$URU^T = \text{diag}(0, N, N, \dots, N) . \quad (6.16)$$

This allows to express the quadratic form in terms of a covariance K diagonal in flavor

$$K := U M U^T \iff M = U^T K U . \quad (6.17)$$

Explicitly K is given by

$$K = \begin{pmatrix} \frac{\pi}{\pi+gN} \frac{1}{-\Delta + \frac{e^2 N}{\pi+gN}} & & & \\ & \frac{1}{-\Delta} & & \\ & & \ddots & \\ & & & \frac{1}{-\Delta} \end{pmatrix} . \quad (6.18)$$

Obviously the covariance K describes one massive and $N-1$ massless particles. Finally the quadratic form reads

$$- \frac{e^2}{2\pi} \left(\varepsilon_{\mu\nu} \partial_\mu U a_\nu, K \varepsilon_{\rho\sigma} \partial_\rho U a_\sigma \right) , \quad (6.19)$$

where matrix notation in flavor space was used.

The next step is to define new sources $A_\mu^{(I)}$ that are linear combinations of the $a_\mu^{(b)}$

$$A_\mu^{(I)} := \sum_{b=1}^N U_{Ib} a_\mu^{(b)} \xleftrightarrow{U^T=U^{-1}} a_\mu^{(b)} = \sum_{I=1}^N U_{Ib} A_\mu^{(I)} . \quad (6.20)$$

Rewriting the coupling term in $E(n_b, n_b; a^{(b)})$ allows the identification of the currents $J_\mu^{(I)}$ that couple to the new sources $A_\mu^{(I)}$

$$\sum_{b=1}^N \left(a_\mu^{(b)}, j_\mu^{(b)} \right) = \sum_{I=1}^N \sum_{b=1}^N \left(U_{Ib} A_\mu^{(I)}, j_\mu^{(b)} \right) := \sum_{I=1}^N \left(A_\mu^{(I)}, J_\mu^{(I)} \right) , \quad (6.21)$$

where I defined

$$J_\mu^{(I)} := \sum_{b=1}^N U_{Ib} j_\mu^{(b)} \quad I = 1, 2, \dots, N . \quad (6.22)$$

Inspecting the explicit form of the matrix U quoted in Appendix B.3 one can express the new currents also as

$$J_\mu^{(I)} := \sum_{b,b'=1}^N \bar{\psi}^{(b)} \gamma_\mu H_{b,b'}^{(I)} \psi^{(b')} \quad (6.23)$$

where the $N \times N$ matrices $H^{(I)}$ are generators of a *Cartan subalgebra* of $U(N)_{\text{flavor}}$

$$\begin{aligned}
H^{(1)} &= \frac{1}{\sqrt{N}} \mathbb{I} , \\
H^{(2)} &= \frac{1}{\sqrt{N-1+(N-1)^2}} \text{diag}(1, 1, \dots, 1, -N+1) , \\
H^{(3)} &= \frac{1}{\sqrt{N-2+(N-2)^2}} \text{diag}(1, 1, \dots, 1, -N+2, 0) , \dots \\
H^{(N)} &= \frac{1}{\sqrt{2}} \text{diag}(1, -1, 0, 0, \dots, 0) .
\end{aligned} \tag{6.24}$$

The currents (6.23) with the generators (6.24) will be referred to as *Cartan type currents*. Note that $J^{(1)}$ is the $U(1)$ -current already defined in (3.12). Later I will also discuss vector currents that correspond to generators that do not belong to this Cartan subalgebra.

Putting things together the generating functional now reads

$$\begin{aligned}
E(n_b, n_b; A^{(I)}) &= \left(\frac{1}{2\pi} \right)^{2 \sum_b n_b} \exp \left(-\frac{e^2}{2\pi} \sum_{I=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} \varepsilon_{\rho\sigma} \partial_\rho A_\sigma^{(I)} \right) \right) \\
&\times \exp \left(2e \sum_{I=1}^N \sum_{b=1}^N \sum_{j=1}^{n_b} U_{Ib} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, \frac{-1}{\Delta} [\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)})] \right) \right) \\
&\times \exp \left(-2e \sum_{I=1}^N \sum_{b,b'=1}^N \sum_{j=1}^{n_b} U_{Ib'} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, \left[\frac{e^2 \pi}{(\pi + gN)^2} \tilde{Q} + \frac{g}{\pi} \tilde{C} \right] [\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)})] \right) \right) \\
&\times \exp \left(2 \sum_{b,b'=1}^N \sum_{j,j'=1}^{n_b, n_{b'}} \left(\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)}) \right), \left[\left(\frac{e\pi}{\pi + gN} \right)^2 \tilde{Q} + g \tilde{C} \right] [\delta_n(x_{j'}^{(b')}) - \delta_n(y_{j'}^{(b')})] \right) \right) \\
&\times \exp \left(-2 \sum_{b=1}^N \left[\sum_{i,j=1}^{n_b} \ln |x_i^{(b)} - y_j^{(b)}| - \frac{1}{2} \sum_{i \neq j}^{n_b} \left[\ln |x_i^{(b)} - x_j^{(b)}| + \ln |y_i^{(b)} - y_j^{(b)}| \right] \right] \right) .
\end{aligned} \tag{6.25}$$

In order to understand the representation of the chiral densities in a bosonized theory one has to study the terms that mix the sources $A_\mu^{(b)}$ and the space-time arguments of the densities. Using (see Appendix B.3, Formula B.17)

$$\sum_{b'=1}^N U_{Ib'} = \frac{1}{\sqrt{N}} N \delta_{I1} = U_{1b} N \delta_{I1} \quad (b \text{ arbitrary}) , \tag{6.26}$$

one can write the part of the exponent that corresponds to the mixing term (linear in $A_\mu^{(b)}$) as

$$2e \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(1)}, \left[\frac{-1}{\Delta} - N \left[\frac{e^2 \pi}{(\pi + gN)^2} \tilde{Q} + \frac{g}{\pi} \tilde{C} \right] \sum_{b=1}^N \sum_{j=1}^{n_b} U_{1b} [\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)})] \right] \right)$$

$$\begin{aligned}
& + 2e \sum_{I=2}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, \frac{-1}{\Delta} \sum_{b=1}^N \sum_{j=1}^{n_b} U_{Ib} [\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)})] \right) \\
& = 2e \sum_{I=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} \sum_{b=1}^N \sum_{j=1}^{n_b} U_{Ib} [\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)})] \right) . \quad (6.27)
\end{aligned}$$

Putting things together one ends up with

$$\begin{aligned}
E(n_b, n_b; A^{(I)}) &= \left(\frac{1}{2\pi} \right)^{2 \sum_b n_b} \times \exp \left(-\frac{e^2}{2\pi} \sum_{I=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} \varepsilon_{\rho\sigma} \partial_\rho A_\sigma^{(I)} \right) \right) \\
&\times \prod_{b=1}^N \prod_{j=1}^{n_b} \exp \left(2e \sum_{I=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} U_{Ib} [\delta_n(x_j^{(b)}) - \delta_n(y_j^{(b)})] \right) \right) \\
&\times \tilde{\rho}_n(\{x_j^{(b)}\}, \{y_j^{(b)}\}) . \quad (6.28)
\end{aligned}$$

$\tilde{\rho}_n(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ denotes the factor that depends on the space-time arguments. Furthermore it still depends on n , the index of the δ -sequence. The wave function renormalization procedure introduced in the last chapter (compare (5.23)) has to be applied before the limit $n \rightarrow \infty$ is taken. I will give the explicit form of $\tilde{\rho}$ after this procedure in the end of this section.

The expression (6.28) now can be generalized to the case

$$n_a - m_a = l \quad l \in \mathbb{Z} \quad a = 1, 2, \dots, N . \quad (6.29)$$

For those cases the θ -prescription gives a nonvanishing result which for $l \neq 0$ is different from the naive expectation functional. The result can be obtained easily by following the argumentation given in the last chapter. In fact the term quadratic in the sources is not affected by the θ -prescription, and $\tilde{\rho}(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ can be read off from (5.48) immediately. Only the term that mixes the sources with the space time arguments of the chiral densities has to be generalized, but this is straightforward. One ends up with

$$\begin{aligned}
& E(n_b, m_b; A^{(b)}) \\
&= \left\langle \prod_{b=1}^N \prod_{i=1}^{n_b} \bar{\psi}^{(b)}(x_i^{(b)}) P_+ \psi^{(b)}(x_i^{(b)}) \prod_{j=1}^{m_b} \bar{\psi}^{(b)}(y_j^{(b)}) P_- \psi^{(b)}(y_j^{(b)}) e^{ie \sum_{b=1}^N (a_\mu^{(b)} \cdot j_\mu^{(b)})} \right\rangle_0^\theta \\
&= \exp \left(-\frac{e^2}{2\pi} \sum_{I=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} \varepsilon_{\rho\sigma} \partial_\rho A_\sigma^{(I)} \right) \right) \\
&\times \prod_{b=1}^N \prod_{j=1}^{n_b} \exp \left(2e \sum_{I=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} U_{Ib} \delta(x_j^{(b)}) \right) \right) \\
&\times \prod_{b=1}^N \prod_{j=1}^{m_b} \exp \left(-2e \sum_{I=1}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K_{II} U_{Ib} \delta(y_j^{(b)}) \right) \right)
\end{aligned}$$

$$\times \rho(\{x_j^{(b)}\}, \{y_j^{(b)}\}) , \quad (6.30)$$

where $\rho(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ is given by (the limit $n \rightarrow \infty$ and the wave function renormalization (compare (5.23)) has already been performed)

$$\begin{aligned} \rho(\{x_j^{(b)}\}, \{y_j^{(b)}\}) = & h(n_b, m_b) \left(\frac{1}{2\pi}\right)^{\sum_b (n_b + m_b)} e^{i\frac{\theta}{N} \sum_b (n_b - m_b)} \times \left(e \sqrt{\frac{N}{\pi + gN}} \frac{e^\gamma}{2} \right)^{\frac{\pi}{\pi + gN} \frac{1}{N} \left(\sum_b (n_b - m_b)\right)^2} \\ & \times \exp \left(\sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_{b'}} \tilde{V}(x_j^{(b)} - y_{j'}^{(b')}) \right) \\ & \times \exp \left(-\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{n_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \tilde{V}(x_j^{(b)} - x_{j'}^{(b')}) \right) \\ & \times \exp \left(-\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \tilde{V}(y_j^{(b)} - y_{j'}^{(b')}) \right) \\ & \times \exp \left(-\sum_{b=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_b} \ln(x_j^{(b)} - y_{j'}^{(b)})^2 \right) \\ & \times \exp \left(\frac{1}{2} \sum_{b=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{n_b} (1 - \delta_{jj'}) \ln(x_j^{(b)} - x_{j'}^{(b)})^2 \right) \\ & \times \exp \left(\frac{1}{2} \sum_{b=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_b} (1 - \delta_{jj'}) \ln(y_j^{(b)} - y_{j'}^{(b)})^2 \right) . \end{aligned} \quad (6.31)$$

The factor $h(n_b, m_b)$ is defined as

$$h(n_b, m_b) := \sum_{l=-\infty}^{+\infty} \prod_{b=1}^N \delta_{n_b - m_b, l} . \quad (6.32)$$

It is equal to one whenever the θ -prescription allows a nonvanishing result for $E(n_b, m_b; A^{(b)})$, otherwise it is zero. \tilde{V} is given by (compare (5.49) and (5.21))

$$\tilde{V}(x) := \frac{1}{N} \ln(x^2) + \frac{2\pi}{N(\pi + gN)} \left(K_0\left(\sqrt{\frac{e^2 N}{\pi + gN}} |x|\right) + \ln\left(\frac{1}{2} \sqrt{\frac{e^2 N}{\pi + gN}}\right) + \gamma \right) . \quad (6.33)$$

Expression (6.30) can now be used to identify the correct bosonization.

6.2 Bosonization prescription

Bosonization means that the generating functional $E(n_b, m_b; A^{(b)})$ can also be obtained by computing the vacuum expectation value of a certain functional $F(n_b, m_b; A^{(b)}; \Phi^{(I)})$ in a bosonic theory with some fields $\Phi^{(I)}$. Every operator that was used to define $E(n_b, m_b; A^{(b)})$ will have a transcription in terms of the $\Phi^{(I)}$ which then enters $F(n_b, m_b; A^{(b)}; \Phi^{(I)})$. Inspecting (6.30) makes it plausible to try it with Gaussian fields $\Phi^{(I)}$ with some covariances $K^{(I)}$ which are related to the K_{II} (see (6.18)). Thus one can express the idea of bosonization in the following formula

$$E(n_b, m_b; A^{(b)}) = \left\langle F(n_b, m_b; A^{(b)}; \Phi^{(I)}) \right\rangle_{\{K^{(I)}\}}, \quad (6.34)$$

where $\langle \dots \rangle_{\{K^{(I)}\}}$ denotes expectation value for the fields $\Phi^{(I)}$ with respect to the covariances $K^{(I)}$. Two steps have to be done. First define an appropriate covariance $K^{(I)}$ and then establish the correct transcription of the fermionic operators into bosonic ones.

The definition of the $K^{(I)}$ is rather simple. I define

$$K^{(1)} := \frac{1}{-\Delta + \frac{e^2 N}{\pi + gN}} = \frac{\pi + gN}{\pi} K_{11}, \quad (6.35)$$

and

$$K^{(I)} := \frac{1}{-\Delta} = K_{II} \quad I = 2, \dots, N. \quad (6.36)$$

Thus the $K^{(I)}$ are just the canonically normalized K_{II} . The term in (6.30) which is quadratic in the sources $A^{(b)}$ then implies the following prescription for the bosonization of the Cartan currents¹

$$J_\nu^{(I)}(x) \longleftrightarrow \begin{cases} -\frac{1}{\sqrt{\pi+gN}} \varepsilon_{\mu\nu} \partial_\mu \Phi^{(1)}(x) & I = 1 \\ -\frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu} \partial_\mu \Phi^{(I)}(x) & I = 2, \dots, N. \end{cases} \quad (6.37)$$

With this choice the term linear in $A^{(b)}$ already fixes the structure of the transcription of the chiral densities to

$$\begin{aligned} & \bar{\psi}^{(b)}(x) P_\pm \psi^{(b)}(x) \longleftrightarrow \\ & \frac{1}{2\pi} c^{(b)} : e^{\mp i 2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x)} :_{M^{(1)}} \prod_{I=2}^N : e^{\mp i 2\sqrt{\pi} U_{Ib} \Phi^{(I)}(x)} :_{M^{(I)}} e^{\pm i \frac{\theta}{N}}, \end{aligned} \quad (6.38)$$

¹The bosonization prescription (6.37) for the Cartan currents was already obtained for the $g = 0$ case in [10].

as can be seen from the exponentials in (6.30) linear in the sources. Here $: \dots :_{M^{(I)}}$ denotes normal ordering with respect to mass $M^{(I)}$ (compare Appendix A.4). Those normal ordering masses as well as the real numbers $c^{(b)}$ are free parameters that will be fixed later.

Inserting the prescriptions (6.37), (6.38) into the definition of $E(n_b, m_b; A^{(b)})$, one obtains

$$\begin{aligned}
E(n_b, m_b; A^{(b)}) &\longleftrightarrow \\
&\left(\frac{1}{2\pi} \right)^{\sum_b (n_b + m_b)} e^{i \frac{\theta}{N} \sum_b (n_b - m_b)} \prod_{b=1}^N \left(c^{(b)} \right)^{n_b + m_b} \\
&\left\langle \prod_{b=1}^N \prod_{j=1}^{n_b} \left[: e^{-i2\sqrt{\pi} \sqrt{\frac{\pi}{\pi + gN}} U_{1b} \Phi^{(1)}(x_j^{(b)})} :_{M^{(1)}} \prod_{I=2}^N : e^{-i2\sqrt{\pi} U_{Ib} \Phi^{(I)}(x_j^{(b)})} :_{M^{(I)}} \right] \right. \\
&\times \prod_{b=1}^N \prod_{j=1}^{m_b} \left[: e^{+i2\sqrt{\pi} \sqrt{\frac{\pi}{\pi + gN}} U_{1b} \Phi^{(1)}(y_j^{(b)})} :_{M^{(1)}} \prod_{I=2}^N : e^{+i2\sqrt{\pi} U_{Ib} \Phi^{(I)}(y_j^{(b)})} :_{M^{(I)}} \right] \\
&\times \exp \left(- \frac{ie}{\sqrt{\pi + gN}} \left(A_\mu^{(1)}, \varepsilon_{\nu\mu} \partial_\nu \Phi^{(1)} \right) - \frac{ie}{\sqrt{\pi}} \sum_{I=2}^N \left(A_\mu^{(I)}, \varepsilon_{\nu\mu} \partial_\nu \Phi^{(I)} \right) \right) \Bigg\rangle_{\{K^{(I)}\}}. \tag{6.39}
\end{aligned}$$

The Gaussian integrals can be solved rather easily since they factorize with respect to the $\Phi^{(I)}$. One then obtains for the right hand side of the last equation

$$\begin{aligned}
&\exp \left(- \frac{e^2}{2\pi} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(1)}, \frac{\pi}{\pi + gN} K^{(1)} \varepsilon_{\rho\sigma} \partial_\rho A_\sigma^{(1)} \right) - \frac{e^2}{2\pi} \sum_{I=2}^N \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K^{(I)} \varepsilon_{\rho\sigma} \partial_\rho A_\sigma^{(I)} \right) \right) \\
&\times \prod_{b=1}^N \prod_{j=1}^{n_b} \exp \left(+ 2e U_{1b} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(1)}, \frac{\pi}{\pi + gN} K^{(1)} \delta(x_j^{(b)}) \right) \right) \\
&\times \prod_{b=1}^N \prod_{j=1}^{n_b} \exp \left(+ 2e \sum_{I=2}^N U_{Ib} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K^{(I)} \delta(x_j^{(b)}) \right) \right) \\
&\times \prod_{b=1}^N \prod_{j=1}^{m_b} \exp \left(- 2e U_{1b} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(1)}, \frac{\pi}{\pi + gN} K^{(1)} \delta(y_j^{(b)}) \right) \right) \\
&\times \prod_{b=1}^N \prod_{j=1}^{m_b} \exp \left(- 2e \sum_{I=2}^N U_{Ib} \left(\varepsilon_{\mu\nu} \partial_\mu A_\nu^{(I)}, K^{(I)} \delta(y_j^{(b)}) \right) \right) \\
&\times \rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\}). \tag{6.40}
\end{aligned}$$

Comparing (6.30) and (6.40) shows immediately that the terms quadratic and linear in the sources come out correctly. Thus it is left to show

$$\rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\}) = \rho(\{x_j^{(b)}\}, \{y_j^{(b)}\}), \tag{6.41}$$

where $\rho(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ is given by (6.31). As mentioned before, the integral over the $\Phi^{(I)}$ factorizes such that $\rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ reads

$$\begin{aligned} \rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\}) &= \left(\frac{1}{2\pi}\right)^{\sum_b (n_b + m_b)} e^{i\frac{\theta}{N} \sum_b (n_b - m_b)} \prod_{b=1}^N \left(c^{(b)}\right)^{n_b + m_b} \\ &\times \left\langle \prod_{b=1}^N \prod_{j=1}^{n_b} : e^{-i2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x_j^{(b)})} :_{M^{(1)}} \prod_{j=1}^{m_b} : e^{+i2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(y_j^{(b)})} :_{M^{(1)}} \right\rangle_{K^{(1)}} \\ &\times \prod_{I=2}^N \left\langle \prod_{b=1}^N \prod_{j=1}^{n_b} : e^{-i2\sqrt{\pi} U_{Ib} \Phi^{(I)}(x_j^{(b)})} :_{M^{(I)}} \prod_{j=1}^{m_b} : e^{+i2\sqrt{\pi} U_{Ib} \Phi^{(I)}(y_j^{(b)})} :_{M^{(I)}} \right\rangle_{K^{(I)}} \end{aligned} \quad (6.42)$$

The $\Phi^{(1)}$ expectation value is rather simple since the covariance is massive. Using $U_{1b} = 1/\sqrt{N}$ one finds (compare Appendix A.4)

$$\begin{aligned} &\left(\frac{e^2 N}{\pi + gN} \frac{1}{(M^{(1)})^2} \right)^{\frac{\pi}{\pi+gN} \frac{1}{2N} \sum_b (n_b + m_b)} \\ &\times \exp \left(\sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_{b'}} 2 \frac{\pi}{\pi + gN} \frac{1}{N} K_0 \left(e \sqrt{\frac{N}{\pi + gN}} |x_j^{(b)} - y_{j'}^{(b')}| \right) \right) \\ &\times \exp \left(- \sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{n_{b'}} \left(1 - \delta_{bb'} \delta_{jj'} \right) \frac{\pi}{\pi + gN} \frac{1}{N} K_0 \left(e \sqrt{\frac{N}{\pi + gN}} |x_j^{(b)} - x_{j'}^{(b')}| \right) \right) \\ &\times \exp \left(- \sum_{b,b'=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_{b'}} \left(1 - \delta_{bb'} \delta_{jj'} \right) \frac{\pi}{\pi + gN} \frac{1}{N} K_0 \left(e \sqrt{\frac{N}{\pi + gN}} |y_j^{(b)} - y_{j'}^{(b')}| \right) \right) . \end{aligned} \quad (6.43)$$

The evaluation of the expectation values $\langle \dots \rangle_{K^{(I)}}$ with $I > 1$ is a little bit more involved, since for massless fields the neutrality condition (see Appendix A.4) is relevant. It will turn out that the neutrality condition produces the factor $h(n_b, m_b)$ which was defined in (6.32). The condition implies

$$\sum_{b=1}^N U_{Ib} (n_b - m_b) \stackrel{!}{=} 0 \quad \forall \quad I = 2, 3, \dots, N, \quad (6.44)$$

for nonvanishing expectation values. Interpreting the lines of U_{Ib} as vectors $\vec{r}^{(I)}$ (see Appendix B.3 for the definition), the condition reads

$$(\vec{n} - \vec{m}) \cdot \vec{r}^{(I)} \stackrel{!}{=} 0 \quad \forall \quad I = 2, 3, \dots, N. \quad (6.45)$$

One finds that the only solution is

$$\vec{n} - \vec{m} \propto (1, 1, \dots, 1). \quad (6.46)$$

Since n_b and m_b are integers this solution is equivalent to multiplication with $h(n_b, m_b)$.

Using Appendix A.4 one obtains

$$\begin{aligned}
& \prod_{I=2}^N \left\langle \prod_{b=1}^N \prod_{j=1}^{n_b} : e^{-i2\sqrt{\pi}U_{Ib}\Phi^{(I)}(x_j^{(b)})} :_{M^{(I)}} \prod_{j=1}^{m_b} : e^{+i2\sqrt{\pi}U_{Ib}\Phi^{(I)}(y_j^{(b)})} :_{M^{(I)}} \right\rangle_{K^{(I)}} \\
&= h(n_b, m_b) \prod_{I=2}^N \left(\frac{1}{M^{(I)}} \right)^{\sum_b (U_{Ib})^2 (n_b - m_b)} \\
&\quad \times \exp \left(- \sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_{b'}} \sum_{I=2}^N U_{Ib} U_{Ib'} \left[\ln(x_j^{(b)} - y_{j'}^{(b')})^2 + 2\gamma - \ln(4) \right] \right) \\
&\quad \times \exp \left(\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{n_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \sum_{I=2}^N U_{Ib} U_{Ib'} \left[\ln(x_j^{(b)} - x_{j'}^{(b')})^2 + 2\gamma - \ln(4) \right] \right) \\
&\quad \times \exp \left(\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \sum_{I=2}^N U_{Ib} U_{Ib'} \left[\ln(y_j^{(b)} - y_{j'}^{(b')})^2 + 2\gamma - \ln(4) \right] \right) \\
&= h(n_b, m_b) \prod_{I=2}^N \left(\frac{1}{M^{(I)}} \right)^{\sum_b (U_{Ib})^2 (n_b - m_b)} \\
&\quad \times \exp \left(\sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_{b'}} \frac{1}{N} \left[\ln(x_j^{(b)} - y_{j'}^{(b')})^2 + 2\gamma - \ln(4) \right] \right) \\
&\quad \times \exp \left(-\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{n_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \frac{1}{N} \left[\ln(x_j^{(b)} - x_{j'}^{(b')})^2 + 2\gamma - \ln(4) \right] \right) \\
&\quad \times \exp \left(-\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \frac{1}{N} \left[\ln(y_j^{(b)} - y_{j'}^{(b')})^2 + 2\gamma - \ln(4) \right] \right) \\
&\quad \times \exp \left(- \sum_{b=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_b} \left[\ln(x_j^{(b)} - y_{j'}^{(b)})^2 + 2\gamma - \ln(4) \right] \right) \\
&\quad \times \exp \left(\frac{1}{2} \sum_{b=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{n_b} (1 - \delta_{jj'}) \left[\ln(x_j^{(b)} - x_{j'}^{(b)})^2 + 2\gamma - \ln(4) \right] \right) \\
&\quad \times \exp \left(\frac{1}{2} \sum_{b=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_b} (1 - \delta_{jj'}) \left[\ln(y_j^{(b)} - y_{j'}^{(b)})^2 + 2\gamma - \ln(4) \right] \right) , \quad (6.47)
\end{aligned}$$

where I made use of (B.20) to remove the U_{Ib} in the last step. Putting things together one ends up with

$$\rho_B(\{x_j^{(b)}\}, \{y_j^{(b)}\})$$

$$\begin{aligned}
&= C \times \exp \left(\sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_{b'}} \tilde{V}(x_j^{(b)} - y_{j'}^{(b')}) \right) \\
&\times \exp \left(-\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \tilde{V}(x_j^{(b)} - x_{j'}^{(b')}) \right) \\
&\times \exp \left(-\frac{1}{2} \sum_{b,b'=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_{b'}} (1 - \delta_{bb'} \delta_{jj'}) \tilde{V}(y_j^{(b)} - y_{j'}^{(b')}) \right) \\
&\times \exp \left(-\sum_{b=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{m_b} \ln(x_j^{(b)} - y_{j'}^{(b)})^2 \right) \\
&\times \exp \left(\frac{1}{2} \sum_{b=1}^N \sum_{j=1}^{n_b} \sum_{j'=1}^{n_b} (1 - \delta_{jj'}) \ln(x_j^{(b)} - x_{j'}^{(b)})^2 \right) \\
&\times \exp \left(\frac{1}{2} \sum_{b=1}^N \sum_{j=1}^{m_b} \sum_{j'=1}^{m_b} (1 - \delta_{jj'}) \ln(y_j^{(b)} - y_{j'}^{(b)})^2 \right). \tag{6.48}
\end{aligned}$$

The constant C is given by

$$\begin{aligned}
C &= h(n_b, m_b) \left(\frac{1}{2\pi} \right)^{\sum_b (n_b + m_b)} e^{i \frac{\theta}{N} \sum_b (n_b - m_b)} \left(e \sqrt{\frac{N}{\pi + gN}} \frac{e^\gamma}{2} \right)^{\frac{\pi}{\pi + gN} \frac{1}{N} (\sum_b (n_b - m_b))^2} \\
&\times \prod_{a=1}^N [c^{(a)}]^{n_a + m_a} \prod_{b=1}^N \left[\left(\frac{2e^{-\gamma}}{M^{(1)}} \right)^{\frac{\pi}{\pi + gN} \frac{1}{N}} \prod_{I=2}^N \left(\frac{2e^{-\gamma}}{M^{(I)}} \right)^{(U_{Ib})^2} \right]^{n_b + m_b}. \tag{6.49}
\end{aligned}$$

The evaluation of the constant C is straightforward but lengthy. One has to add to the Bessel functions and the logarithms in the exponent terms proportional to

$$\frac{\pi}{\pi + gN} \left[\ln \left(\frac{e}{2} \sqrt{\frac{N}{\pi + gN}} \right) + \gamma \right] \tag{6.50}$$

to obtain \tilde{V} (see (6.33)). This modification has to be compensated by a factor that enters C . The factor picks up an exponent proportional to $(\sum_b (n_b - m_b))^2$, which shows up also in the result for $\rho(\{x_j^{(b)}\}, \{y_j^{(b)}\})$ (see (6.31)). It is rather crucial since it cannot be produced by the factors $c^{(b)}$ of the ansatz (6.38), which can only obtain exponents linear in n_b and m_b . Furthermore one has to use $U_{1b} = 1/\sqrt{N}$ and (B.20) to obtain (6.49). Finally (6.41) can be fulfilled by setting

$$c^{(b)} = \left(\frac{M^{(1)} e^\gamma}{2} \right)^{\frac{\pi}{\pi + gN} \frac{1}{N}} \prod_{I=2}^N \left(\frac{M^{(I)} e^\gamma}{2} \right)^{(U_{Ib})^2}. \tag{6.51}$$

Thus the bosonization is given by (6.37) and (6.38) together with (6.51).

I finish this section with a discussion of the Cartan currents in the massless model. Up to a constant the vector currents $J^{(I)}$ are bosonized by $-\frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu} \partial_\mu \Phi^{(I)}$. The covariance $K^{(I)}$ (6.35) then implies that the U(1)-current $J^{(1)}$ describes a particle with mass

$$e \sqrt{\frac{N}{\pi + gN}}. \quad (6.52)$$

The rest of the Cartan currents are massless as can be seen from the covariances $K(I)$, $I = 2, 3, \dots, N$.

It has to be remarked that the U(1)-current remains massive also in the case $g = 0$. At first glance it might seem a little bit suspicious that the U(1)-current which is treated differently from the others (only $J^{(1)}$ has the Thirring term in the action), acquires mass. As long as one is not interested in the massive theory, the Thirring term is not needed, and g can be set equal to zero since the expectation values are continuous in g . (6.52) shows that $J^{(1)}$ remains massive.

6.3 More vector currents

In this section the vector currents that are not of the Cartan type will be discussed. In analogy to the construction of meson states in QCD, one can define vector currents for all the generators of U(N). A convenient basis of the Lie algebra of U(N) is given by the $N(N-1)/2$ generators $H^{(I)}$ of the form

$$\frac{1}{\sqrt{2}} \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix}, \quad (6.53)$$

and $N(N-1)/2$ of the form

$$\frac{1}{\sqrt{2}} \begin{pmatrix} & & & -i \\ & & & \\ & & & \\ i & & & \end{pmatrix}. \quad (6.54)$$

The new generators are given indices $I = N+1, N+2, \dots, N^2$. Together with $H^{(1)}, \dots, H^{(N)}$ defined in (6.24) they generate U(N)². The corresponding vector

² It has to be remarked that the generators (6.24), (6.53) and (6.54) I use are not normalized as usual (like e.g. the Gell-Mann matrices for SU(3)). This is due to the fact,

currents are (compare (6.23))

$$J_\mu^{(I)}(x) := \sum_{b,b'=1}^N \bar{\psi}^{(b)}(x) \gamma_\mu H_{b,b'}^{(I)} \psi^{(b')}(x) . \quad (6.55)$$

Since only different flavors (which cannot contract) sit at one space-time point no short distance singularity can emerge. For the Cartan currents $J_\mu^{(1)}, \dots, J_\mu^{(N)}$ (which are diagonal in flavor) this problem was circumvented by including the sources that couple to these currents into the fermion determinant which has its own renormalization.

First I notice that the set of the N^2 vector currents generate orthogonal states

$$\langle J_\mu^{(I)}(x) J_\nu^{(I')}(y) \rangle = \delta_{II'} \mathcal{F}_{\mu\nu}^{(I)}(x, y) \quad I = 1, 2, \dots, N^2 , \quad (6.56)$$

where \mathcal{F} is the two point function. For the Cartan currents (6.56) follows directly from the bosonization. To prove it for the set of all N^2 currents one has to take functional derivatives of the generating functional (3.38) with respect to the sources $\eta, \bar{\eta}$ and $A_\mu^{(I)} \Leftrightarrow a_\mu^{(b)}$. If $I \neq I'$, either the different flavors do not contract entirely, or terms with opposite sign cancel.

The two point function for the new currents can be obtained easily. In the case $I = I', I = N+1, N+2, \dots, N^2$, functional derivation of (3.38) leads to (take e.g. $\mu = \nu = 1$)

$$\begin{aligned} \mathcal{F}_{11} = & - \int d\mu_{\tilde{Q}}[\varphi] d\mu_{\tilde{C}}[\theta] \left(G_{21}(x, y; \varphi, \theta) G_{21}(y, x; \varphi, \theta) + \right. \\ & \left. G_{12}(x, y; \varphi, \theta) G_{12}(y, x; \varphi, \theta) \right) . \end{aligned} \quad (6.57)$$

Using the explicit form (4.58) for G , one immediately sees that the exponentials involving φ and θ cancel. Integration over φ simply gives a factor 1. The same is true for arbitrary μ, ν . Hence the two point function is the same as for currents made from free, massless fermions. It can be expressed in terms of derivatives of the propagator of a free massless boson, giving rise to the same expression as was obtained for the Cartan type currents, i.e. for $I = 2, 3, \dots, N$.

Putting things together one concludes for the two-point functions of the vector currents

$$\langle J_\mu^{(I)}(x) J_\nu^{(I')}(y) \rangle = \delta_{II'} \begin{cases} \frac{1}{\pi+gN} \varepsilon_{\mu\rho} \partial x_\rho \varepsilon_{\nu\sigma} \partial y_\sigma \langle \varphi^{(I)}(x) \varphi^{(I)}(y) \rangle_{K^{(1)}} & I = 1 \\ \frac{1}{\pi} \varepsilon_{\mu\rho} \partial x_\rho \varepsilon_{\nu\sigma} \partial y_\sigma \langle \varphi^{(I)}(x) \varphi^{(I)}(y) \rangle_{K^{(I)}} & I = 2, \dots, N, \end{cases} \quad (6.58)$$

that the generators (6.24) stem from the orthonormal matrix U which diagonalizes the covariance (compare (6.17)).

where the scalar field $\varphi^{(1)}$ has the mass given in equation (6.52) and the $\varphi^{(I)}$, $I = 2, 3, \dots, N^2$ are massless.

The N^2 vector have the same mass as the corresponding Bose fields. Only the $U(1)$ -current is massive, while the others are massless. When the model is considered as a toy model for QCD, the vector currents describe the pseudoscalar mesons. Note that $J_{\mu 5}^{(I)} = -i\varepsilon_{\mu\nu} J_{\nu}^{(I)}$ due to the choice of the γ -matrices (compare Appendix B.1). The particle related to $J^{(1)}$ has to be identified with the η' meson. Since it is the only massive current, the model perfectly mimics the $U(1)$ -problem and its solution.

The lesson that has to be learned here reads

Lesson 2 :

The axial $U(1)$ -symmetry is not a symmetry on the physical Hilbert space, and there is no $U(1)$ -problem.

This can be seen rather easily in the $N = 2$ flavor case. The Lagrangian for the scalar fields $\varphi^{(1)}$, $\varphi^{(2)}$ that bosonize the currents and the chiral densities then, is given by

$$\frac{1}{2}(\partial_{\mu}\varphi^{(1)})^2 + \frac{1}{2}(\partial_{\mu}\varphi^{(2)})^2 + \frac{1}{2}(\varphi^{(1)})^2 \frac{e^2 N}{\pi + gN} . \quad (6.59)$$

The bosonization prescription (6.38) for the left-handed densities gives

$$\begin{aligned} \bar{\psi}^{(1)}(x)P_+\psi^{(1)}(x) &\longleftrightarrow \frac{1}{2\pi}c^{(1)} : e^{-ia\varphi^{(1)}(x)} :_{M^{(1)}} : e^{-ib\varphi^{(2)}(x)} :_{M^{(2)}} e^{i\frac{\theta}{2}} , \\ \bar{\psi}^{(2)}(x)P_+\psi^{(2)}(x) &\longleftrightarrow \frac{1}{2\pi}c^{(2)} : e^{-ia\varphi^{(1)}(x)} :_{M^{(1)}} : e^{+ib\varphi^{(2)}(x)} :_{M^{(2)}} e^{i\frac{\theta}{2}} , \end{aligned} \quad (6.60)$$

with (see B.17) $a = 2\sqrt{\pi/2}\sqrt{\pi/(\pi + gN)}$ and $b = 2\sqrt{\pi/2}$. The axial transformation (3.16) acts on the densities via

$$\bar{\psi}^{(b)}(x)P_+\psi^{(b)}(x) \longrightarrow \bar{\psi}^{(b)}(x)P_+\psi^{(b)}(x) e^{i2\omega} . \quad (6.61)$$

In the bosonized theory this corresponds to

$$\begin{aligned} a\varphi^{(1)}(x) + b\varphi^{(2)}(x) &\longrightarrow a\varphi^{(1)}(x) + b\varphi^{(2)}(x) - 2\omega , \\ a\varphi^{(1)}(x) - b\varphi^{(2)}(x) &\longrightarrow a\varphi^{(1)}(x) - b\varphi^{(2)}(x) - 2\omega . \end{aligned} \quad (6.62)$$

Obviously this is not a symmetry, since ω on the right hand side of (6.62) cannot be transformed away, by shifting one of the fields by a constant. $\varphi^{(1)}(x)$ cannot be shifted since it is massive (see (6.59)). $\varphi^{(2)}(x)$ would have to be shifted by $+2\omega/b$ in order to remove ω in the first line of (6.62) and by $-2\omega/b$ to remove it in the second line. Thus $U(1)_A$ is not a symmetry, and no Goldstone particle exists in the physical Hilbert space. The generalization of the arguments to $N > 2$ flavors is straightforward.

6.4 Theorems on n-point functions

To gain more insight into the sector generated by the vector currents in the massless model, I compute explicit expressions for connected n -point functions. Considering fully connected correlations has the advantage of excluding manifestly contractions of fermions at the same point. Therefore one can avoid using test functions as would be required in principle by the distributional nature of the fields. Let

$$\begin{aligned}
C_{n+1} &:= \left\langle J_{\mu_0}^{(I_0)}(x_0) J_{\mu_1}^{(I_1)}(x_1) \dots J_{\mu_n}^{(I_n)}(x_n) \right\rangle_{0\ c}^\theta \\
&= \sum_{a_i, b_i} H_{a_0 b_0}^{(I_0)} H_{a_1 b_1}^{(I_1)} \dots H_{a_n b_n}^{(I_n)} \sum_{\alpha_i, \beta_i} (\gamma_{\mu_0})_{\alpha_0 \beta_0} (\gamma_{\mu_1})_{\alpha_1 \beta_1} \dots (\gamma_{\mu_n})_{\alpha_n \beta_n} \\
&\quad \left\langle \bar{\psi}_{\alpha_0}^{(a_0)}(x_0) \psi_{\beta_0}^{(b_0)}(x_0) \bar{\psi}_{\alpha_1}^{(a_1)}(x_1) \psi_{\beta_1}^{(b_1)}(x_1) \dots \bar{\psi}_{\alpha_n}^{(a_n)}(x_n) \psi_{\beta_n}^{(b_n)}(x_n) \right\rangle_{0\ c}^\theta, \quad (6.63)
\end{aligned}$$

where (6.23) was inserted. Nonvanishing contributions occur only if the color indices can form a closed chain (e.g. $b_0 = a_1, b_1 = a_2, \dots, b_{n-1} = a_n, b_n = a_0$), such that the fermions can contract entirely. The corresponding factor is simply the trace over the flavor matrices $H^{(I)}$. To find all possible contributions one has to sum over all permutations π , keeping the first term fixed.

$$\begin{aligned}
C_{n+1} &= - \sum_{\pi(1,2,\dots,n)} \text{Tr} \left[H^{(I_0)} H^{(I_{\pi(1)})} \dots H^{(I_{\pi(n)})} \right] \sum_{\alpha_i, \beta_i} (\gamma_{\mu_0})_{\alpha_0 \beta_0} (\gamma_{\mu_1})_{\alpha_1 \beta_1} \dots (\gamma_{\mu_n})_{\alpha_n \beta_n} \\
&\quad \int d\mu_{\tilde{Q}}[\varphi] d\mu_{\tilde{C}}[\theta] G_{\beta_0 \alpha_{\pi(1)}}(x_0, x_{\pi(1)}; \varphi, \theta) G_{\beta_{\pi(1)} \alpha_{\pi(2)}}(x_{\pi(1)}, x_{\pi(2)}; \varphi, \theta) \dots \\
&\quad \dots G_{\beta_{\pi(n)} \alpha_0}(x_{\pi(n)}, x_0; \varphi, \theta) \quad (6.64)
\end{aligned}$$

Since in the chosen representation the γ -matrices and the propagator G have only off-diagonal entries (cf. Equation(4.58)), one finds the following chain of implications for e.g. $\beta_0 = 1$

$$\beta_0 = 1 \Rightarrow \alpha_{\pi(1)} = 2 \Rightarrow \beta_{\pi(1)} = 1 \Rightarrow \alpha_{\pi(2)} = 2 \dots \Rightarrow \beta_{\pi(n)} = 1 \Rightarrow \alpha_0 = 2. \quad (6.65)$$

When starting with $\beta_0 = 2$ one ends up with the reverse result of all $\beta_i = 2$ and all $\alpha_i = 1$. Besides those two no other nonvanishing terms contribute. Again one finds that all dependence on φ and θ cancels, and only free propagators G^0 remain. Using $G^0(x) = \frac{1}{2\pi} \frac{\gamma_\mu x_\mu}{x^2}$ and making use of the complex notation (A.15) one obtains

$$\begin{aligned}
C_{n+1} &= - \frac{1}{(2\pi)^{n+1}} \sum_{\pi(1,2,\dots,n)} \text{Tr} \left[H^{(I_0)} H^{(I_{\pi(1)})} \dots H^{(I_{\pi(n)})} \right] \\
&\quad \left\{ \prod_{i=0}^n (\gamma_{\mu_i})_{21} \frac{1}{\tilde{x}_0 - \tilde{x}_{\pi(1)}} \frac{1}{\tilde{x}_{\pi(1)} - \tilde{x}_{\pi(2)}} \dots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_0} + c.c. \right\}, \quad (6.66)
\end{aligned}$$

where *c.c.* denotes complex conjugation.

Using this basic formula, I construct for $N \geq 2$ and arbitrary n fully connected n -point functions that do not vanish; in other words it will be shown that the entire set of all N^2 currents is not Gaussian. For simplicity the construction is performed for the case of $N = 2$. Since for arbitrary $N > 2$ there exist generators with the same commutation relations, it is obvious how to generalize the construction to general N . For $N = 2$ the generators needed in the construction are simply the Pauli matrices up to a normalization factor. To distinguish them from arbitrary generators $H^{(I)}$ (which would be there for $N > 2$), I denote the special set of matrices needed in the construction

$$\tau^{(1)} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^{(2)} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^{(3)} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.67)$$

Define

$$F_{n+1}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n) := \sum_{\pi(1,2,\dots,n)} \text{Tr}[\tau^{(I_0)} \tau^{(I_{\pi(1)})} \dots \tau^{(I_{\pi(n)})}] \frac{1}{\tilde{x}_0 - \tilde{x}_{\pi(1)}} \frac{1}{\tilde{x}_{\pi(1)} - \tilde{x}_{\pi(2)}} \dots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_0}, \quad (6.68)$$

where the set of τ matrices is given by

$$\{\tau^{(I_0)}, \tau^{(I_1)}, \dots, \tau^{(I_n)}\} := \begin{cases} \{\tau^{(2)}, \tau^{(3)}, \tau^{(3)}, \dots, \tau^{(3)}, \tau^{(1)}\} & \text{for } n+1 = 2m+1 \quad (n+1 \text{ odd}) \\ \{\tau^{(1)}, \tau^{(3)}, \tau^{(3)}, \dots, \tau^{(3)}, \tau^{(1)}\} & \text{for } n+1 = 2m+2 \quad (n+1 \text{ even}) \end{cases}. \quad (6.69)$$

Theorem 6.1 :

For arbitrary $m \geq 1$:

$$F_{2m+1} \neq 0, \quad F_{2m+2} \neq 0. \quad (6.70)$$

Proof:

The statement will be proven by induction.

i: $F_3 \neq 0$ and $F_4 \neq 0$ can be checked easily.

ii: Assume $F_{2m+1} \neq 0$ and $F_{2m+2} \neq 0$.

iii: One has to show $F_{2m+3} \neq 0$ and $F_{2m+4} \neq 0$. First I check F_{2m+3} . Again I abbreviate $2m+3 := n+1$. The trick is to consider $F_{n+1}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n)$ as a function of \tilde{x}_0 , and to compute the corresponding residues³.

$$\text{Res}_{\tilde{x}_0}[F_{n+1}, \tilde{x}_0 = \tilde{x}_1]$$

³ I acknowledge a useful discussion with Peter Breitenlohner on this point.

$$\begin{aligned}
&= \sum_{\pi(1,2,\dots,n), \pi(1)=1} \text{Tr} \left[\tau^{(2)} \tau^{(3)} \tau^{(\pi(2))} \dots \tau^{(\pi(n))} \right] \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi(2)}} \frac{1}{\tilde{x}_{\pi(2)} - \tilde{x}_{\pi(3)}} \dots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_1} \\
&- \sum_{\pi'(1,2,\dots,n), \pi'(n)=1} \text{Tr} \left[\tau^{(2)} \tau^{(\pi'(1))} \dots \tau^{(\pi'(n-1))} \tau^{(3)} \right] \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi'(1)}} \frac{1}{\tilde{x}_{\pi'(1)} - \tilde{x}_{\pi'(2)}} \dots \frac{1}{\tilde{x}_{\pi'(n-1)} - \tilde{x}_1}
\end{aligned} \tag{6.71}$$

For a given permutation π choose the permutation π' such that $\pi'(1) = \pi(2), \pi'(2) = \pi(3), \dots, \pi'(n-1) = \pi(n), \pi'(n) = \pi(1) = 1$. This gives for the residue

$$\begin{aligned}
&\sum_{\pi(1,2,\dots,n), \pi(1)=1} \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi(2)}} \frac{1}{\tilde{x}_{\pi(2)} - \tilde{x}_{\pi(3)}} \dots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_1} \\
&\left\{ \text{Tr} \left[\tau^{(2)} \tau^{(3)} \tau^{(\pi(2))} \tau^{(\pi(3))} \dots \tau^{(\pi(n))} \right] - \text{Tr} \left[\tau^{(2)} \tau^{(\pi(2))} \tau^{(\pi(3))} \dots \tau^{(\pi(n))} \tau^{(3)} \right] \right\} . \tag{6.72}
\end{aligned}$$

Using the cyclicity of the trace and $\{\tau^{(2)}, \tau^{(3)}\} = 0$, one finds that the second trace is the negative of the first one. Furthermore $\tau^{(2)} \tau^{(3)} = \frac{i}{\sqrt{2}} \tau^{(1)}$. Looking at the definition (6.69) of F_n for even n one concludes

$$\text{Res}_{\tilde{x}_0} [F_{n+1}, \tilde{x}_0 = \tilde{x}_1] = i\sqrt{2} F_n(\tilde{x}_1, \dots, \tilde{x}_n) = i\sqrt{2} F_{2m+2}(\tilde{x}_1, \dots, \tilde{x}_{2m+2}) . \tag{6.73}$$

Since by assumption $F_{2m+2} \neq 0$, $\text{Res}[F_{2m+3}] \neq 0$ and hence F_{2m+3} does not vanish. The same trick can be applied to prove that this implies $F_{2m+4} \neq 0$. \square

A $n+1$ -point function with the flavor content given by (6.69) is simply the real part of a multiple of F_{n+1} (compare (6.66)). Hence there exist nonvanishing, fully connected n -point functions for arbitrary n .

This result implies that it is not possible to bosonize the whole set of all N^2 vector currents $J_\mu^{(I)}$, $I = 1, 2, \dots, N^2$ linearly in terms of free bosons (this was the reason why Witten [70] introduced his *nonabelian bosonization*). The bosonization sketched above (*abelian bosonization*) can be done only for a Cartan subalgebra where all generators commute, and the two traces in Equation (6.71) cancel. The last observation allows to prove a second theorem.

Theorem 6.2 :

Any fully connected n -point function

$$\left\langle J_{\mu_0}^{(1)}(x_0) J_{\mu_1}^{(I_1)}(x_1) J_{\mu_2}^{(I_2)}(x_2) \dots J_{\mu_n}^{(I_n)}(x_n) \right\rangle_{0c}^\theta \tag{6.74}$$

containing the $U(1)$ -current vanishes, with the exception of the two point function $\langle J_\mu^{(1)}(x) J_\nu^{(1)}(y) \rangle_{0c}^\theta$ (without loss of generality I shifted the $U(1)$ -current in Equation (6.74) to the first position).

Proof :

To prove the statement I use the same trick as for Theorem 6.1. Again I consider the residues of functions F_{n+1} defined by

$$F_{n+1}(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n) :=$$

$$\sum_{\pi(1,2,\dots,n)} \text{Tr} \left[H^{(1)} H^{(I_{\pi(1)})} \dots H^{(I_{\pi(n)})} \right] \frac{1}{\tilde{x}_0 - \tilde{x}_{\pi(1)}} \frac{1}{\tilde{x}_{\pi(1)} - \tilde{x}_{\pi(2)}} \dots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_0} . \quad (6.75)$$

For the residue at $\tilde{x}_0 = \tilde{x}_1$ one obtains an equation equivalent to (6.71). After making the same choice for π' one finds

$$\text{Res}_{\tilde{x}_0} [F_{n+1}, \tilde{x}_0 = \tilde{x}_1] = \sum_{\pi(1,2,\dots,n), \pi(1)=1} \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi(2)}} \frac{1}{\tilde{x}_{\pi(2)} - \tilde{x}_{\pi(3)}} \dots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_1} \\ \left\{ \text{Tr} \left[H^{(1)} H^{(I_1)} H^{(I_{\pi(2)})} \dots H^{(I_{\pi(n)})} \right] - \text{Tr} \left[H^{(1)} H^{(I_{\pi(2)})} \dots H^{(I_{\pi(n)})} H^{(I_1)} \right] \right\} . \quad (6.76)$$

Since $H^{(1)}$ commutes with all generators $H^{(I)}$, the two traces are the same and cancel. Using the same argument one can show that all residues at $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ vanish. So F_{n+1} is analytic and bounded in the entire \tilde{x}_0 plane. By Liouville's theorem F_{n+1} is a constant, and the limit $\tilde{x}_0 \rightarrow \infty$ shows that this constant is zero. Since the $n+1$ -point function defined in (6.74) is proportional to F_{n+1} , it has to vanish. \square

Theorem 6.2 allows to prove the following proposition about the structure of the Hilbert space.

Proposition 6.1 :

The Hilbert space \mathcal{H} generated by the vector currents from the vacuum is the tensor product

$$\mathcal{H} = \mathcal{H}_{U(1)} \otimes \mathcal{H}_{\text{mass}=0} . \quad (6.77)$$

Proof :

To prove this one uses the connection between untruncated and fully connected n -point functions (see e.g. [31]).

$$\langle \phi_1 \dots \phi_n \rangle = \sum_{\pi \in \mathcal{P}_n} \prod_{p \in \pi} \langle \phi_{i_1} \dots \phi_{i_{|p|}} \rangle_c , \quad (6.78)$$

where \mathcal{P}_n is the set of all partitions of $\{1, 2, \dots, n\}$, $\pi = \{p_1, p_2, \dots, p_{|\pi|}\}$ denotes an element of \mathcal{P}_n , and $\{i_1, i_2, \dots, i_{|p|}\}$ is an element p of π . An arbitrary $n+k$ -point function (without loss of generality I write the U(1)-currents first; $I_i \neq 1, i = 1, 2, \dots, k$) factorizes due to Theorem 6.2

$$\begin{aligned} & \left\langle J_{\mu_1}^{(1)}(x_1) J_{\mu_2}^{(1)}(x_2) \dots J_{\mu_n}^{(1)}(x_n) J_{\nu_1}^{(I_1)}(y_1) J_{\nu_2}^{(I_2)}(y_2) \dots J_{\nu_k}^{(I_k)}(y_k) \right\rangle_0^\theta \\ &= \sum_{\pi \in \mathcal{P}_{n+k}} \prod_{p \in \pi} \left\langle J_{\mu_{i_1}}^{(1)}(x_{i_1}) J_{\mu_{i_2}}^{(1)}(x_{i_2}) \dots J_{\mu_{i_{|p|}}}^{(1)}(x_{i_{|p|}}) J_{\nu_{j_1}}^{(I_{j_1})}(y_{j_1}) J_{\nu_{j_2}}^{(I_{j_2})}(y_{j_2}) \dots J_{\nu_{j_{|p|}}}^{(I_{j_k})}(y_{j_{|p|}}) \right\rangle_{0c}^\theta \\ &= \left[\sum_{\pi \in \mathcal{P}_n} \prod_{p \in \pi} \left\langle J_{\mu_{i_1}}^{(1)}(x_{i_1}) J_{\mu_{i_2}}^{(1)}(x_{i_2}) \dots J_{\mu_{i_{|p|}}}^{(1)}(x_{i_{|p|}}) \right\rangle_{0c}^\theta \right] \end{aligned}$$

$$\begin{aligned}
& \times \left[\sum_{\pi' \in \mathcal{P}_k} \prod_{p' \in \pi'} \left\langle J_{\nu_{j_1}}^{(I_{j_1})}(y_{j_1}) J_{\nu_{j_2}}^{(I_{j_2})}(y_{j_2}) \dots J_{\nu_{j_{|p'|}}}^{(I_{j_k})}(y_{j_{|p'|}}) \right\rangle_{0 \ c}^\theta \right] \\
& = \left\langle J_{\mu_1}^{(1)}(x_1) J_{\mu_2}^{(1)}(x_2) \dots J_{\mu_n}^{(1)}(x_n) \right\rangle \left\langle J_{\nu_1}^{(I_1)}(y_1) J_{\nu_2}^{(I_2)}(y_2) \dots J_{\nu_k}^{(I_k)}(y_k) \right\rangle_0^\theta. \quad (6.79)
\end{aligned}$$

From this the tensor product structure of the Hilbert space follows easily. \square

It was pointed out in Equations (6.64), (6.65) that the dependence on the gauge field of the fully connected correlations cancels entirely. Hence the vector currents in the 'massless' sector of the Hilbert space obey the same algebra as in a system of uncoupled fermions; this is the well-known level 1 representation of the $SU(N)_L \times SU(N)_R$ current (Kac-Moody) algebra (see for instance [32]).

Chapter 7

The generalized Sine Gordon model

The Cartan currents can be bosonized in the massive model as well. This gives rise to a generalized Sine Gordon model that will be identified in the first section of this chapter. Furthermore I will show that the expansion in terms of the fermion masses converges if a space-time cutoff is introduced. It will be argued in Section 7.3 that the known methods to remove the cutoff fail. The spectrum of the model will be discussed semiclassically in Section 7.4, and a Witten-Veneziano formula will be shown to hold in 7.5 in this approximation.

7.1 Definition of the model

In Chapter 6 it was shown that it is possible to find a common bosonization of the vector currents together with the chiral densities which show up in the mass perturbation series. The task of this section is to identify the bosonic model in which the Cartan currents of the massive model are bosonized by the prescription (6.37). For the one flavor case ($N = 1$) this is provided by the Sine Gordon model [20]. Thus for $N > 1$ one expects a generalization.

As has been outlined in Chapter 3.4 the strategy for the construction of the massive model is to sum up the expansion of the mass term of the action. If one is interested in generating functionals for Cartan currents, the formula corresponding to (3.37) reads

$$\begin{aligned} F(A_\mu^{(I)}) &:= \frac{1}{Z} \left\langle \exp \left(-S_M[\bar{\psi}, \psi] \right) \exp \left(ie \sum_{I=1}^N \left(A_\mu^{(I)}, J_\mu^{(I)} \right) \right) \right\rangle_0^\theta \\ &= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \left(S_M[\bar{\psi}, \psi] \right)^n \exp \left(ie \sum_{I=1}^N \left(A_\mu^{(I)}, J_\mu^{(I)} \right) \right) \right\rangle_0^\theta \\ &= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(\sum_{b=1}^N m^{(b)} \int_\Lambda d^2x \, t(x) \left[\bar{\psi}^{(b)}(x) P_+ \psi^{(b)}(x) + \bar{\psi}^{(b)}(x) P_- \psi^{(b)}(x) \right] \right)^n \right\rangle \end{aligned}$$

$$\exp \left(i e \sum_{I=1}^N \left(A_\mu^{(I)}, J_\mu^{(I)} \right) \right) \Bigg\rangle_0^\theta. \quad (7.1)$$

Of course also Z has to be expanded in that way. Obviously all the expansion coefficients are of the form of the generalized generating functional $E(n_b, m_b; A^{(b)})$ (c.f. (6.1)) which was entirely bosonized in Section 6.2. Thus one can insert the bosonization prescriptions (6.37), (6.38) and obtain

$$\begin{aligned} F(A_\mu^{(I)}) &= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(\frac{1}{2\pi} \sum_{b=1}^N m^{(b)} c^{(b)} \int_{\Lambda} d^2x \, t(x) \right. \right. \\ &\quad \left[: e^{-i2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x)} :_{M^{(1)}} \prod_{I=2}^N : e^{-i2\sqrt{\pi} U_{Ib} \Phi^{(I)}(x)} :_{M^{(I)}} e^{+i\frac{\theta}{N}} \right. \\ &\quad \left. \left. + : e^{+i2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x)} :_{M^{(1)}} \prod_{I=2}^N : e^{+i2\sqrt{\pi} U_{Ib} \Phi^{(I)}(x)} :_{M^{(I)}} e^{-i\frac{\theta}{N}} \right] \right\rangle^n \\ &\exp \left(-i e \left[\frac{1}{\sqrt{\pi+gN}} \left(A_\mu^{(1)}, \varepsilon_{\nu\mu} \partial_\nu \Phi^{(1)} \right) + \sum_{I=2}^N \frac{1}{\sqrt{\pi}} \left(A_\mu^{(I)}, \varepsilon_{\nu\mu} \partial_\nu \Phi^{(I)} \right) \right] \right) \Bigg\rangle_{\{K^{(I)}\}} = \\ &\left\langle \exp \left(-S_{int}[\Phi^{(I)}] - i e \left[\frac{1}{\sqrt{\pi+gN}} \left(A_\mu^{(1)}, \varepsilon_{\nu\mu} \partial_\nu \Phi^{(1)} \right) + \sum_{I=2}^N \frac{1}{\sqrt{\pi}} \left(A_\mu^{(I)}, \varepsilon_{\nu\mu} \partial_\nu \Phi^{(I)} \right) \right] \right) \right\rangle_{\{K^{(I)}\}}. \end{aligned} \quad (7.2)$$

In the last step I summed up the formal expansion in the quark masses in the bosonized model. The convergence of the series will be proven in the next section justifying the expansion. The part of the action $S_{int}[\Phi^{(I)}]$ that describes the interaction of the fermions can be read off as

$$\begin{aligned} S_{int}[\Phi^{(I)}] &:= -\frac{1}{\pi} \sum_{b=1}^N m^{(b)} c^{(b)} \int_{\Lambda} d^2x \, t(x) \\ &: \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x) + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi^{(I)}(x) - \frac{\theta}{N} \right) :. \end{aligned} \quad (7.3)$$

The Wick ordering of the cosine is understood in the way it is defined in the perturbation expansion (7.2).

I conclude this section with displaying the classical Lagrangian \mathcal{L}_{GSG} which corresponds to the newly defined model. It can be read off from the $K^{(I)}$ (see (6.35) and (6.36)) and S_{int}

$$\begin{aligned} \mathcal{L}_{GSG} &= +\frac{1}{2} \sum_{I=1}^N \partial_\mu \Phi^{(I)} \partial_\mu \Phi^{(I)} + \frac{1}{2} \left(\Phi^{(1)} \right)^2 \frac{e^2 N}{\pi + gN} \\ &- \frac{1}{\pi} \sum_{b=1}^N m^{(b)} c^{(b)} \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)} + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi^{(I)} - \frac{\theta}{N} \right). \end{aligned} \quad (7.4)$$

The model which is described by this Lagrangian will be referred to as the *Generalized Sine Gordon model* (GSG).

At this point I draw another lesson that recovers a property of the θ -vacuum in QCD.

Lesson 3 :

Physics does not depend on θ if at least one of the fermion masses vanishes.

This property of the QCD θ -vacuum was discussed in Section 2.2. In QED₂ it can be seen by the following arguments.

Without loss of generality $m^{(2)}$ can be set to zero. Using

$$U_{N1} = \frac{1}{\sqrt{2}} \quad , \quad U_{N2} = \frac{-1}{\sqrt{2}} \quad , \quad U_{Nb} = 0 \quad \text{for } 3 \leq b \leq N \quad , \quad (7.5)$$

for $n \geq 2$ (compare (B.17)), one obtains for the interaction term (7.3)

$$\begin{aligned} & \sum_{b=1}^N m^{(b)} c^{(b)} : \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x) + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi^{(I)}(x) - \frac{\theta}{N} \right) : \\ &= m^{(1)} c^{(1)} : \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x) + 2\sqrt{\pi} \sum_{I=2}^{N-1} U_{Ib} \Phi^{(I)}(x) \right. \\ & \quad \left. + 2\sqrt{\pi} \frac{1}{\sqrt{2}} \Phi^{(N)}(x) - \frac{\theta}{N} \right) : \\ &+ \sum_{b=3}^N m^{(b)} c^{(b)} : \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1b} \Phi^{(1)}(x) + 2\sqrt{\pi} \sum_{I=2}^{N-1} U_{Ib} \Phi^{(I)}(x) - \frac{\theta}{N} \right) : . \end{aligned} \quad (7.6)$$

Since $\Phi^{(N)}$ is a massless field and shows up only in the first term on the right hand side of (7.6) it can be shifted by a constant in order to change θ . If none of the masses vanishes, $\Phi^{(N)}$ enters the interaction term twice but with different sign, as can be seen from (7.5). The value of θ cannot be changed then, and physics depends on it.

7.2 Convergence of the mass perturbation series

In this section it is proven that the mass perturbation series converges if a space-time cutoff Λ is imposed. The proof follows mainly the strategy developed by Fröhlich for the one flavor case [25], [26]. At several points some extra work has to be done, and it turns out that the equations for more than one flavor become rather monstrous. In other words, this section is of a more technical nature.

The interaction term under consideration can be rewritten as (compare (7.3))

$$U_\Lambda := -2 \sum_{b=1}^N \beta^{(b)} \int_\Lambda d^2x \, t(x) : \cos \left(2\sqrt{\pi} \sum_{I=1}^N U_{Ib} \Phi^{(I)}(x) - \frac{\theta}{N} \right) :_{K^W} . \quad (7.7)$$

Λ denotes a finite rectangle in \mathbb{R}^2 and t is some test function with $\sup_{x \in \Lambda} t(x) \leq 1$. $\beta^{(b)}$ are real positive coefficients (compare (7.3), (6.51)). Wick ordering is with respect to

$$K^W := \text{diag} \left(\frac{\pi}{\pi + gN} \frac{1}{-\Delta + \frac{e^2 N}{\pi + gN}}, \frac{1}{-\Delta + 1}, \dots, \frac{1}{-\Delta + 1} \right) . \quad (7.8)$$

The covariance for the expectation values reads

$$K^\mu := \text{diag} \left(\frac{\pi}{\pi + gN} \frac{1}{-\Delta + \frac{e^2 N}{\pi + gN}}, \frac{1}{-\Delta + \mu}, \dots, \frac{1}{-\Delta + \mu} \right) , \quad (7.9)$$

and the limit $\mu \rightarrow 0$ is taken in the end. Note the factor $\pi/(\pi + gN)$ in front of the first entry of K^μ and K^W . It was included in the covariance in order to remove $\sqrt{\pi/(\pi + gN)}$ from the first argument of the cosine (compare (7.3) and (7.7)). It will be proven

$$e^{-U_\Lambda} \text{ is integrable with respect to } d\mu_{K^0}[\Phi] ,$$

$$\text{and the mass perturbation series converges.} \quad (7.10)$$

where the covariance K^0 formally denotes the limit $\mu \rightarrow 0$ taken in the end, and the exponential function is understood in the sense of its expansion. In particular one has to prove that the series converges. Since the proof is rather lengthy, I decided to divide it into several steps.

Step 1 : Convenient notation

The diagonal covariance (7.9) can easily be interpreted from a quantum field theoretical point of view, but is less suitable for the proof below. Define new fields $\varphi^{(b)}$, $b = 1, 2, \dots, N$ by

$$\varphi^{(b)} := \sum_{I=1}^N U_{Ib} \Phi^{(I)} . \quad (7.11)$$

Using the orthogonality of the matrix U (compare Appendix B.3) one immediately finds that (7.10) is equivalent to showing that

$$\mathcal{E} := \exp \left(2 \sum_{b=1}^N \beta^{(b)} \int_\Lambda d^2x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) - \frac{\theta}{N} \right) :_{Q^W} \right) , \quad (7.12)$$

is integrable with respect to $d\mu_{Q^0}[\varphi]$ where

$$Q_{ab}^\mu := \sum_{I=1}^N U_{Ia} U_{Ib} K_{II}^\mu , \quad (7.13)$$

and

$$Q_{ab}^W := \sum_{I=1}^N U_{Ia} U_{Ib} K_{II}^W . \quad (7.14)$$

Step 2 : Dependence on θ

Define

$$\chi_\pm^{(b)}[C] := \int_\Lambda d^2x \, t(x) : e^{\pm i 2\sqrt{\pi} \varphi^{(b)}(x)} :_C , \quad (7.15)$$

and

$$\sigma_\pm[C] := \sum_{a=1}^N \beta^{(a)} \chi_\pm^{(a)}[C] , \quad (7.16)$$

where C is some covariance. Thus

$$\begin{aligned} \mathcal{E} &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left[e^{-i\frac{\theta}{N}} \sigma_+[Q^W] + e^{+i\frac{\theta}{N}} \sigma_-[Q^W] \right]^n \right\rangle_{Q^\mu} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{q=0}^n \binom{n}{q} e^{-i\frac{\theta}{N}q} e^{+i\frac{\theta}{N}(n-q)} \left\langle \left[\sigma_+[Q^W] \right]^q \left[\sigma_-[Q^W] \right]^{n-q} \right\rangle_{Q^\mu} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{2} \left\{ \sum_{q=0}^n \binom{n}{q} e^{-i\frac{\theta}{N}2q} e^{+i\frac{\theta}{N}(n-q)} \left\langle \left[\sigma_+[Q^W] \right]^q \left[\sigma_-[Q^W] \right]^{n-q} \right\rangle_{Q^\mu} \right. \\ &\quad \left. + \sum_{r=0}^n \binom{n}{n-r} e^{-i\frac{\theta}{N}r} e^{+i\frac{\theta}{N}(n-r)} \left\langle \left[\sigma_+[Q^W] \right]^r \left[\sigma_-[Q^W] \right]^{n-r} \right\rangle_{Q^\mu} \right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{q=0}^n \binom{n}{q} \cos\left(\frac{\theta}{N}(n-2q)\right) \left\langle \left[\sigma_+[Q^W] \right]^q \left[\sigma_-[Q^W] \right]^{n-q} \right\rangle_{Q^\mu} . \quad (7.17) \end{aligned}$$

In the last step I made use of $\binom{n}{n-r} = \binom{n}{r}$, performed a transformation of the summation index $q := n - r$ in the second sum, and applied

$$\left\langle \left[\sigma_+[Q^W] \right]^q \left[\sigma_-[Q^W] \right]^{n-q} \right\rangle_{Q^\mu} = \left\langle \left[\sigma_+[Q^W] \right]^{n-q} \left[\sigma_-[Q^W] \right]^q \right\rangle_{Q^\mu} . \quad (7.18)$$

The latter can be seen to hold from a transformation $\varphi \longrightarrow -\varphi$ which transforms $\sigma_\pm[Q^W] \longrightarrow \sigma_\mp[Q^W]$ but leaves the expectation value invariant. Thus \mathcal{E} can always be bounded by the $\theta = 0$ expression

$$\left\langle \exp \left(2 \sum_{b=1}^N \beta^{(b)} \int_\Lambda d^2x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) \right) :_{Q^W} \right) \right\rangle . \quad (7.19)$$

All the expectation values involved are real, since the coefficients $\beta^{(b)}$ and also the Gaussian integrals (see (A.21)) are real.

Step 3 : cosh-bound

The following estimate can be seen to hold if the cosh is expressed in terms of exponentials

$$\begin{aligned}
& \left\langle \exp \left(2 \sum_{b=1}^N \beta^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) \right) :_{Q^W} \right) \right\rangle_{Q^\mu} \\
& \leq \left\langle \prod_{b=1}^N 2 \cosh \left(2\beta^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) \right) :_{Q^W} \right) \right\rangle_{Q^\mu} \\
& := \left\langle \prod_{b=1}^N 2 \left(\sum_{n_b=0}^{\infty} \frac{1}{(2n_b)!} (\beta^{(b)})^{2n_b} \left[\chi_+^{(b)}[Q^W] + \chi_-^{(b)}[Q^W] \right]^{2n_b} \right) \right\rangle_{Q^\mu} \\
& = 2^N \prod_{a=1}^N \sum_{n_a=0}^{\infty} \frac{(\beta^{(a)})^{2n_a}}{(2n_a)!} \prod_{b=1}^N \sum_{q_b=0}^{2n_b} \binom{2n_b}{q_b} E^\mu(\{q, n\}) , \tag{7.20}
\end{aligned}$$

where

$$E^\mu(\{q, n\}) := \left\langle \prod_{b=1}^N \left[\chi_+^{(b)}[Q^W] \right]^{q_b} \left[\chi_-^{(b)}[Q^W] \right]^{2n_b - q_b} \right\rangle_{Q^\mu} . \tag{7.21}$$

Step 4 : Change of Wick ordering

Change of Wick ordering means (compare Appendix A.4)

$$: e^{i2\sqrt{\pi} \varphi^{(b)}(x)} :_{Q^{old}} = : e^{i2\sqrt{\pi} \varphi^{(b)}(x)} :_{Q^{new}} \lim_{z \rightarrow 0} e^{\frac{1}{2}4\pi [Q_{bb}^{old}(z) - Q_{bb}^{new}(z)]} . \tag{7.22}$$

Now I set

$$Q^{old} := Q^W , \quad Q^{new} := Q^\mu . \tag{7.23}$$

One has to evaluate

$$\begin{aligned}
Q_{bb}^{old}(z) - Q_{bb}^{new}(z) &= \sum_{I=1}^N U_{Ib} U_{Ib} \left[K_{II}^W(z) - K_{II}^\mu(z) \right] \\
&= \sum_{I=1}^N U_{Ib} U_{Ib} (1 - \delta_{I1}) \left[\frac{1}{4\pi} \ln(\mu^2) + O(z^2) \right] , \tag{7.24}
\end{aligned}$$

where I used (7.8), (7.9) and (A.43) in the last step. Thus one obtains

$$: e^{i2\sqrt{\pi} \varphi^{(b)}(x)} :_{Q^W} = : e^{i2\sqrt{\pi} \varphi^{(b)}(x)} :_{Q^\mu} (\mu)^{\sum_{I=2}^N (U_{Ib})^2} . \tag{7.25}$$

Inserting this in (7.21) one ends up with

$$E^\mu(\{q, n\}) = \prod_{a=1}^N (\mu)^{2n_a \sum_{I=2}^N (U_{Ia})^2} \left\langle \prod_{b=1}^N [\chi_+^{(b)}[Q^\mu]]^{q_b} [\chi_-^{(b)}[Q^\mu]]^{2n_b - q_b} \right\rangle_{Q^\mu}. \quad (7.26)$$

Step 5 : Inverse conditioning

In this step the inverse conditioning formula (B.22) from Appendix B.4 is applied for

$$C^1 := Q^\mu \quad , \quad C^2 := Q^W. \quad (7.27)$$

Thus one has to check $C^1 \geq C^2$. Obviously

$$\begin{aligned} (f, [C^1 - C^2] f) &= (f, U^T [K^\mu - K^W] U f) \\ &= (g, \text{diag}(0, 1, \dots, 1) \left[\frac{1}{-\Delta + \mu^2} - \frac{1}{-\Delta + 1} \right] g) \geq 0, \end{aligned} \quad (7.28)$$

for $\mu^2 \leq 1$ and inverse conditioning can be applied. g was defined as $g = Uf$ and f denotes an arbitrary vector of test functions. Formula (B.22) gives

$$\begin{aligned} &\left\langle \prod_{b=1}^N [\chi_+^{(b)}[Q^\mu]]^{q_b} [\chi_-^{(b)}[Q^\mu]]^{2n_b - q_b} \right\rangle_{Q^\mu} \\ &\leq \left\langle \prod_{b=1}^N [\chi_+^{(b)}[Q^W]]^{q_b} [\chi_-^{(b)}[Q^W]]^{2n_b - q_b} \right\rangle_{Q^W} e^{4\pi \sum_{b=1}^N 2n_b \lambda^{(b)}}. \end{aligned} \quad (7.29)$$

for $\mu^2 \leq 1$. One has to compute

$$\begin{aligned} \lambda^{(b)} &= \lim_{z \rightarrow 0} \frac{1}{2} [C_{bb}^1(z) - C_{bb}^2(z)] = \lim_{z \rightarrow 0} \frac{1}{2} \sum_{I=1}^N (U_{Ib})^2 [K_{II}^\mu(z) - K_{II}^W(z)] \\ &= -\frac{1}{2} \sum_{I=2}^N (U_{Ib})^2 \frac{1}{4\pi} \ln(\mu^2), \end{aligned} \quad (7.30)$$

where (7.27), (7.13), (7.14) and (A.43) from the appendix were used in the last step. Thus the power of μ that emerges from the change of Wick ordering in (7.26) is eaten up by the power that comes from inverse conditioning, as can be seen from (7.29). Thus the combination of changing the Wick ordering and inverse conditioning leads to an upper bound for \mathcal{E} which does not depend on μ any longer

$$\left\langle \prod_{b=1}^N 2 \cosh \left(2\beta^{(b)} \int_{\Lambda} d^2x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) \right) :_{Q^W} \right) \right\rangle_{Q^W}$$

$$= 2^N \prod_{a=1}^N \sum_{n_a=0}^{\infty} \frac{(\beta^{(a)})^{2n_a}}{(2n_a)!} \prod_{b=1}^N \sum_{q_b=0}^{2n_b} \binom{2n_b}{q_b} \left\langle \prod_{c=1}^N [\chi_+^{(c)}[Q^W]]^{q_c} [\chi_-^{(c)}[Q^W]]^{2n_c-q_c} \right\rangle_{Q^W}. \quad (7.31)$$

Step 6 : Conditioning

Define $M^2 := \min(1, e^2 N/(\pi + gN))$ which implies

$$\frac{1}{-\Delta + M^2} \geq \frac{1}{-\Delta + \frac{e^2}{\pi + gN}} \quad , \quad \frac{1}{-\Delta + M^2} \geq \frac{1}{-\Delta + 1}. \quad (7.32)$$

Define

$$K^M := \frac{1}{-\Delta + M^2} \text{diag} \left(\frac{\pi}{\pi + gN}, 1, \dots, 1 \right), \quad (7.33)$$

and

$$Q^M := U^T K^M U = \frac{1}{-\Delta + M^2} \alpha. \quad (7.34)$$

The matrix α is defined as (use (B.20) in the second step)

$$\alpha_{ab} := \frac{\pi}{\pi + gN} U_{1a} U_{1b} + \sum_{I=2}^N U_{Ia} U_{Ib} = \frac{\pi}{\pi + gN} \frac{1}{N} + \delta_{ab} - \frac{1}{N} = \delta_{ab} - \frac{g}{\pi + gN}. \quad (7.35)$$

Inspecting (7.32) and (7.34) immediately shows (the argument is similar to (7.28)) that $Q^M \geq Q^W$. Thus Corollary B.1 (Equation (B.30) in the conditioning appendix) can be applied for $C^1 = Q^M$ and $C^2 = Q^W$, and the upper bound (7.31) is replaced by

$$\begin{aligned} & \left\langle \prod_{b=1}^N 2 \cosh \left(2\beta^{(b)} \int_{\Lambda} d^2x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) \right) :_{Q^M} \right) \right\rangle_{Q^M} \\ &= 2^N \prod_{a=1}^N \sum_{n_a=0}^{\infty} \frac{(\beta^{(a)})^{2n_a}}{(2n_a)!} \prod_{b=1}^N \sum_{q_b=0}^{2n_b} \binom{2n_b}{q_b} \left\langle \prod_{c=1}^N [\chi_+^{(c)}[Q^M]]^{q_c} [\chi_-^{(c)}[Q^M]]^{2n_c-q_c} \right\rangle_{Q^M}. \end{aligned} \quad (7.36)$$

Step 7 : Dirichlet boundary conditions

Define a new covariance

$$Q_{ab}^{M,S} := \frac{1}{-\Delta_S + M^2} \alpha_{ab}, \quad (7.37)$$

where Δ_S is the Laplace operator with zero Dirichlet data on the circle ∂S , as is discussed in Appendix B.6. Using (B.34) and Formula (7.37) one immediately infers

$$Q^M \geq Q^{M,S}, \quad (7.38)$$

and the inverse conditioning formula (B.22) can be applied.

$$\begin{aligned} \lambda^{(b)} &= \lim_{x \rightarrow y} \frac{1}{2} \left[Q_{bb}^M(x, y) - Q_{bb}^{M,S}(x, y) \right] = \\ &\left(1 - \frac{g}{\pi + gN} \right) \lim_{x \rightarrow y} \frac{1}{2} \left[\frac{1}{-\Delta + M^2}(x, y) - \frac{1}{-\Delta_S + M^2}(x, y) \right] \leq \left(1 - \frac{g}{\pi + gN} \right) \tilde{\omega} . \end{aligned} \quad (7.39)$$

In the first step the explicit form (7.35) of α_{ab} was used. In the second step the bound (B.35) was included. Defining now

$$\omega := \exp \left(4\pi \left(1 - \frac{g}{\pi + gN} \right) \tilde{\omega} \right) \quad (7.40)$$

one concludes from the inverse conditioning formula

$$\begin{aligned} &\left\langle \prod_{b=1}^N [\chi_+^{(b)} [Q^M]]^{q_b} [\chi_-^{(b)} [Q^M]]^{2n_b - q_b} \right\rangle_{Q^M} \\ &\leq \omega^{\sum_{a=1}^N 2n_a} \left\langle \prod_{b=1}^N [\chi_+^{(b)} [Q^{M,S}]]^{q_b} [\chi_-^{(b)} [Q^{M,S}]]^{2n_b - q_b} \right\rangle_{Q^{M,S}} . \end{aligned} \quad (7.41)$$

Inserting this into (7.36) the upper bound for \mathcal{E} now reads

$$\left\langle \prod_{b=1}^N 2 \cosh \left(2\omega \beta^{(b)} \int_{\Lambda} d^2x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) \right) :_{Q^{M,S}} \right) \right\rangle_{Q^{M,S}} . \quad (7.42)$$

Using (B.33) one concludes (similar to (7.28))

$$Q^{0,S} \geq Q^{M,S} . \quad (7.43)$$

Conditioning then gives the new upper bound

$$\begin{aligned} &\left\langle \prod_{b=1}^N 2 \cosh \left(2\omega \beta^{(b)} \int_{\Lambda} d^2x \, t(x) : \cos \left(2\sqrt{\pi} \varphi^{(b)}(x) \right) :_{Q^{0,S}} \right) \right\rangle_{Q^{0,S}} = \\ &2^N \prod_{a=1}^N \sum_{n_a=0}^{\infty} \frac{(\omega \beta^{(a)})^{2n_a}}{(2n_a)!} \prod_{b=1}^N \sum_{q_b=0}^{2n_b} \binom{2n_b}{q_b} \left\langle \prod_{c=1}^N [\chi_+^{(c)} [Q^{0,S}]]^{q_c} [\chi_-^{(c)} [Q^{0,S}]]^{2n_c - q_c} \right\rangle_{Q^{0,S}} . \end{aligned} \quad (7.44)$$

Step 8 : Reduction to neutral contributions

Since $\chi_+^{(b)} [C] = \overline{\chi_-^{(b)} [C]}$ one can bound the expectation values in (7.44) by their neutral contributions :

$$\left\langle \prod_{b=1}^N [\chi_+^{(b)} [Q^{0,S}]]^{q_b} [\chi_-^{(b)} [Q^{0,S}]]^{2n_b - q_b} \right\rangle_{Q^{0,M}}$$

$$\leq \left\langle \prod_{b=1}^N \left| \chi_+^{(b)}[Q^{0,S}] \right|^{q_b} \left| \chi_-^{(b)}[Q^{0,S}] \right|^{2n_b - q_b} \right\rangle_{Q^{0,S}} = \left\langle \prod_{b=1}^N \left[\chi_+^{(b)}[Q^{0,S}] \chi_-^{(b)}[Q^{0,S}] \right]^{n_b} \right\rangle_{Q^{0,S}}. \quad (7.45)$$

Using

$$\sum_{q_b=1}^{2n_b} \binom{2n_b}{q_b} = 2^{2n_b}, \quad (7.46)$$

one ends up with (insert (7.45) in (7.44))

$$\mathcal{E} \leq 2^N \prod_{a=1}^N \sum_{n_a=0}^{\infty} \frac{(2\omega\beta^{(a)})^{2n_a}}{(2n_a)!} \left\langle \prod_{b=1}^N \left[\chi_+^{(b)}[Q^{0,S}] \chi_-^{(b)}[Q^{0,S}] \right]^{n_b} \right\rangle_{Q^{0,S}}. \quad (7.47)$$

Step 9 : Explicit evaluation

$$\begin{aligned} & \left\langle \prod_{b=1}^N \left[\chi_+^{(b)}[Q^{0,S}] \chi_-^{(b)}[Q^{0,S}] \right]^{n_b} \right\rangle_{Q^{0,S}} \\ &= \int_{\Lambda} \prod_{a=1}^N \left[\prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} t(x_i^{(a)}) t(y_i^{(a)}) \right] \\ & \times \left\langle \prod_{b=1}^N \prod_{j=1}^{n_b} \left[: e^{+i2\sqrt{\pi}\varphi^{(b)}(x_j^{(b)})} :_{Q^{0,S}} : e^{-i2\sqrt{\pi}\varphi^{(b)}(y_j^{(b)})} :_{Q^{0,S}} \right] \right\rangle_{Q^{0,S}} \leq \\ & \int_{\Lambda} \prod_{a=1}^N \left[\prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \right] \left\langle \prod_{b=1}^N \prod_{j=1}^{n_b} \left[: e^{+i2\sqrt{\pi}\varphi^{(b)}(x_j^{(b)})} :_{Q^{0,S}} : e^{-i2\sqrt{\pi}\varphi^{(b)}(y_j^{(b)})} :_{Q^{0,S}} \right] \right\rangle_{Q^{0,S}} \\ &= \int_{\Lambda} \prod_{a=1}^N \left[\prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \right] \exp \left\{ -4\pi \frac{1}{2} \sum_{b,c=1}^N \sum_{j=1}^{n_b} \sum_{l=1}^{n_c} \left[(1 - \delta_{bc} \delta_{jl}) \right. \right. \\ & \times \left. \left(Q_{bc}^{0,S}(x_j^{(b)}, x_l^{(c)}) + Q_{bc}^{0,S}(y_j^{(b)}, y_l^{(c)}) \right) - Q_{bc}^{0,S}(x_j^{(b)}, y_l^{(c)}) - Q_{bc}^{0,S}(y_j^{(b)}, x_l^{(c)}) \right] \Big\} \\ &= \int_{\Lambda} \prod_{a=1}^N \left[\prod_i^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \right] \\ & \times \prod_{b=1}^N e^{-2\pi \left(1 - \frac{g}{\pi+gN}\right) \sum_{j=1}^{n_b} \sum_{l=1}^{n_b} \left[(1 - \delta_{jl}) (C^{0,S}(x_j^{(b)}, x_l^{(b)}) + C^{0,S}(y_j^{(b)}, y_l^{(b)})) - 2C^{0,S}(x_j^{(b)}, y_l^{(b)}) \right]} \\ & \times \prod_{\substack{c=1 \\ c \neq d}}^N e^{2\pi \frac{g}{\pi+gN} \sum_{k=1}^{n_c} \sum_{h=1}^{n_d} \left[C^{0,S}(x_k^{(c)}, x_h^{(d)}) + C^{0,S}(y_k^{(c)}, y_h^{(d)}) - C^{0,S}(x_k^{(c)}, y_h^{(d)}) - C^{0,S}(y_k^{(c)}, x_h^{(d)}) \right]}. \end{aligned} \quad (7.48)$$

In the first step $\sup_{x \in \Lambda} t(x) \leq 1$ was used to remove the test function t . Then the Gaussian integral was solved, and finally $Q_{ab}^{0,S}(x, y) = C^{0,S}(x, y) \alpha_{ab}$

(compare (7.37) and (7.35)) was inserted. Using the explicit form (B.36) for the propagator $C^{0,S}$ one obtains

$$\begin{aligned}
& e^{-2\pi\left(1-\frac{g}{\pi+gN}\right)\sum_{i=1}^{n_a}\sum_{j=1}^{n_a}\left[(1-\delta_{ij})(C^{0,S}(x_i^{(a)},x_j^{(a)})+C^{0,S}(y_i^{(a)},y_j^{(a)})-2C^{0,S}(x_i^{(a)},y_j^{(a)})\right]} \\
&= \left| \frac{\prod_{i<j}^{n_a} (\tilde{x}_i^{(a)} - \tilde{x}_j^{(a)})(\tilde{y}_i^{(a)} - \tilde{y}_j^{(a)})(\hat{x}_i^{(a)} - \hat{x}_j^{(a)})(\hat{y}_i^{(a)} - \hat{y}_j^{(a)})}{\prod_{i,j=1}^{n_a} (\tilde{x}_i^{(a)} - \tilde{y}_j^{(a)})(\hat{x}_i^{(a)} - \hat{y}_j^{(a)})} \right|^{[1-\frac{g}{\pi+gN}]} \\
&\times \left| \frac{\prod_{i,j=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{y}_j^{(a)})(\tilde{x}_i^{(a)} - \hat{y}_j^{(a)})}{\prod_{i,j=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{x}_j^{(a)})(\tilde{y}_i^{(a)} - \hat{y}_j^{(a)})} \right|^{[1-\frac{g}{\pi+gN}]} \times \left| \prod_{i=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{x}_i^{(a)})(\tilde{y}_i^{(a)} - \hat{y}_i^{(a)}) \right|^{[1-\frac{g}{\pi+gN}]} .
\end{aligned} \tag{7.49}$$

For the definition of \tilde{x} and \hat{x} see (B.37) and (B.38). The third factor on the right hand side can be bounded due to the geometrical setting (see Figure B.1)

$$\left| \prod_{i=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{x}_i^{(a)})(\tilde{y}_i^{(a)} - \hat{y}_i^{(a)}) \right|^{[1-\frac{g}{\pi+gN}]} \leq \left(8^{[1-\frac{g}{\pi+gN}]}\right)^{2n_a} , \tag{7.50}$$

where I made use of (B.39). Evaluating explicitly the last term in (7.48) gives ($a \neq b$)

$$\begin{aligned}
& e^{2\pi\frac{g}{\pi+gN}\sum_{i=1}^{n_a}\sum_{j=1}^{n_b}\left[C^{0,S}(x_i^{(a)},x_j^{(b)})+C^{0,S}(y_i^{(a)},y_j^{(b)})-C^{0,S}(x_i^{(a)},y_j^{(b)})-C^{0,S}(y_i^{(a)},x_j^{(b)})\right]} \\
&= \left| \frac{\prod_{i=1}^{n_a}\prod_{j=1}^{n_b} (\tilde{x}_i^{(a)} - \tilde{y}_j^{(b)})(\tilde{y}_i^{(a)} - \tilde{x}_j^{(b)})(\hat{x}_i^{(a)} - \hat{y}_j^{(b)})(\hat{y}_i^{(a)} - \hat{x}_j^{(b)})}{\prod_{i=1}^{n_a}\prod_{j=1}^{n_b} (\tilde{x}_i^{(a)} - \tilde{x}_j^{(b)})(\tilde{y}_i^{(a)} - \tilde{y}_j^{(b)})(\hat{x}_i^{(a)} - \hat{x}_j^{(b)})(\hat{y}_i^{(a)} - \hat{y}_j^{(b)})} \right|^{\frac{1}{2}\frac{g}{\pi+gN}} \\
&\times \left| \frac{\prod_{i=1}^{n_a}\prod_{j=1}^{n_b} (\hat{x}_i^{(a)} - \tilde{x}_j^{(b)})(\hat{y}_i^{(a)} - \tilde{y}_j^{(b)})(\tilde{x}_i^{(a)} - \hat{x}_j^{(b)})(\tilde{y}_i^{(a)} - \hat{y}_j^{(b)})}{\prod_{i=1}^{n_a}\prod_{j=1}^{n_b} (\hat{x}_i^{(a)} - \tilde{y}_j^{(b)})(\hat{y}_i^{(a)} - \tilde{x}_j^{(b)})(\tilde{x}_i^{(a)} - \hat{y}_j^{(b)})(\tilde{y}_i^{(a)} - \hat{x}_j^{(b)})} \right|^{\frac{1}{2}\frac{g}{\pi+gN}} .
\end{aligned} \tag{7.51}$$

Putting things together (use (7.48)-(7.51))

$$\begin{aligned}
& \left\langle \prod_{b=1}^N \left[\chi_+^{(b)} [Q^{0,S}] \chi_-^{(b)} [Q^{0,S}] \right]^{n_b} \right\rangle_{Q^{0,S}} \\
&\leq \prod_{a=1}^N \left(8^{[1-\frac{g}{\pi+gN}]}\right)^{2n_a} \int_{\Lambda} \prod_{b=1}^N \prod_{i=1}^{n_b} d^2x_i^{(b)} d^2y_i^{(b)} \mathcal{F}(\{x, y\}) ,
\end{aligned} \tag{7.52}$$

where I defined

$$\begin{aligned}
\mathcal{F}(\{x, y\}) &:= \prod_{a=1}^N \left| \frac{\prod_{i,j=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{y}_j^{(a)})(\tilde{x}_i^{(a)} - \hat{y}_j^{(a)})}{\prod_{i,j=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{x}_j^{(a)})(\tilde{y}_i^{(a)} - \hat{y}_j^{(a)})} \right|^{[1-\frac{g}{\pi+gN}]} \\
&\times \prod_{a=1}^N \left| \frac{\prod_{i<j}^{n_a} (\tilde{x}_i^{(a)} - \tilde{x}_j^{(a)})(\tilde{y}_i^{(a)} - \tilde{y}_j^{(a)})(\hat{x}_i^{(a)} - \hat{x}_j^{(a)})(\hat{y}_i^{(a)} - \hat{y}_j^{(a)})}{\prod_{i,j=1}^{n_a} (\tilde{x}_i^{(a)} - \tilde{y}_j^{(a)})(\hat{x}_i^{(a)} - \hat{y}_j^{(a)})} \right|^{[1-\frac{g}{\pi+gN}]}
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{a \neq b}^N \left| \frac{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\tilde{x}_i^{(a)} - \tilde{y}_j^{(b)})(\tilde{y}_i^{(a)} - \tilde{x}_j^{(b)})(\hat{x}_i^{(a)} - \hat{y}_j^{(b)})(\hat{y}_i^{(a)} - \hat{x}_j^{(b)})}{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\tilde{x}_i^{(a)} - \tilde{x}_j^{(b)})(\tilde{y}_i^{(a)} - \tilde{y}_j^{(b)})(\hat{x}_i^{(a)} - \hat{x}_j^{(b)})(\hat{y}_i^{(a)} - \hat{y}_j^{(b)})} \right|^{\frac{1}{2} \frac{g}{\pi + gN}} \\
& \times \prod_{a \neq b}^N \left| \frac{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\hat{x}_i^{(a)} - \tilde{x}_j^{(b)})(\hat{y}_i^{(a)} - \tilde{y}_j^{(b)})(\tilde{x}_i^{(a)} - \hat{x}_j^{(b)})(\tilde{y}_i^{(a)} - \hat{y}_j^{(b)})}{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\hat{x}_i^{(a)} - \tilde{x}_j^{(b)})(\hat{y}_i^{(a)} - \tilde{y}_j^{(b)})(\tilde{x}_i^{(a)} - \hat{x}_j^{(b)})(\tilde{y}_i^{(a)} - \hat{y}_j^{(b)})} \right|^{\frac{1}{2} \frac{g}{\pi + gN}}.
\end{aligned} \tag{7.53}$$

Step 10 : Cauchy's identity

Define the following vectors of complex space-time arguments

$$\begin{aligned}
w^{(a)} &:= \begin{pmatrix} \tilde{x}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{x}_{n^a}^{(a)} \\ \hat{y}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \hat{y}_{n^a}^{(a)} \end{pmatrix}, \quad z^{(a)} := \begin{pmatrix} \tilde{y}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}_{n^a}^{(a)} \\ \hat{x}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_{n^a}^{(a)} \end{pmatrix}, \quad w^{(a,b)} := \begin{pmatrix} \tilde{x}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{x}_{n^a}^{(a)} \\ \hat{y}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \hat{y}_{n^a}^{(a)} \\ \hat{x}_1^{(b)} \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_{n^b}^{(b)} \\ \tilde{y}_1^{(b)} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}_{n^b}^{(b)} \end{pmatrix}, \quad z^{(a,b)} := \begin{pmatrix} \tilde{y}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}_{n^a}^{(a)} \\ \hat{x}_1^{(a)} \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_{n^a}^{(a)} \\ \hat{y}_1^{(b)} \\ \cdot \\ \cdot \\ \cdot \\ \hat{y}_{n^b}^{(b)} \\ \tilde{x}_1^{(b)} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{x}_{n^b}^{(b)} \end{pmatrix}.
\end{aligned} \tag{7.54}$$

Cauchy's identity (see [22], compare (5.27))

$$\left| \det_{(i,j)} \left(\frac{1}{w_i - z_j} \right) \right| = \left| \frac{\prod_{1 \leq i < j \leq n} (w_i - w_j)(z_i - z_j)}{\prod_{i,j=1}^n (w_i - z_j)} \right|, \tag{7.55}$$

can be used to rewrite the integrand \mathcal{F} in terms of determinants. A straightforward computation gives

$$\left| \det_{i,j=1,\dots,2n_a} \left(\frac{1}{w_i^{(a)} - z_j^{(a)}} \right) \right|$$

$$\begin{aligned}
&= \left| \frac{\prod_{i < j}^{n_a} (\tilde{x}_i^{(a)} - \tilde{x}_j^{(a)})(\tilde{y}_i^{(a)} - \tilde{y}_j^{(a)})(\hat{x}_i^{(a)} - \hat{x}_j^{(a)})(\hat{y}_i^{(a)} - \hat{y}_j^{(a)})}{\prod_{i,j=1}^{n_a} (\tilde{x}_i^{(a)} - \tilde{y}_j^{(a)})(\hat{x}_i^{(a)} - \hat{y}_j^{(a)})} \right| \\
&\quad \times \left| \frac{\prod_{i,j=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{y}_j^{(a)})(\tilde{x}_i^{(a)} - \hat{y}_j^{(a)})}{\prod_{i,j=1}^{n_a} (\hat{x}_i^{(a)} - \tilde{x}_j^{(a)})(\tilde{y}_i^{(a)} - \hat{y}_j^{(a)})} \right|, \tag{7.56}
\end{aligned}$$

and ($a \neq b$)

$$\begin{aligned}
&\left| \det_{i,j=1,\dots,2(n_a+n_b)} \left(\frac{1}{w_i^{(a,b)} - z_j^{(a,b)}} \right) \right| \\
&= \left| \det_{i,j=1,\dots,2n_a} \left(\frac{1}{w_i^{(a)} - z_j^{(a)}} \right) \right| \times \left| \det_{i,j=1,\dots,2n_b} \left(\frac{1}{w_i^{(b)} - z_j^{(b)}} \right) \right| \\
&\quad \times \left| \frac{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\tilde{x}_i^{(a)} - \tilde{y}_j^{(b)})(\tilde{y}_i^{(a)} - \tilde{x}_j^{(b)})(\hat{x}_i^{(a)} - \hat{y}_j^{(b)})(\hat{y}_i^{(a)} - \hat{x}_j^{(b)})}{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\tilde{x}_i^{(a)} - \tilde{x}_j^{(b)})(\tilde{y}_i^{(a)} - \tilde{y}_j^{(b)})(\hat{x}_i^{(a)} - \hat{x}_j^{(b)})(\hat{y}_i^{(a)} - \hat{y}_j^{(b)})} \right| \\
&\quad \times \left| \frac{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\hat{x}_i^{(a)} - \tilde{x}_j^{(b)})(\hat{y}_i^{(a)} - \tilde{y}_j^{(b)})(\tilde{x}_i^{(a)} - \hat{x}_j^{(b)})(\tilde{y}_i^{(a)} - \hat{y}_j^{(b)})}{\prod_{i=1}^{n_a} \prod_{j=1}^{n_b} (\hat{x}_i^{(a)} - \tilde{y}_j^{(b)})(\hat{y}_i^{(a)} - \tilde{x}_j^{(b)})(\tilde{x}_i^{(a)} - \hat{y}_j^{(b)})(\tilde{y}_i^{(a)} - \hat{x}_j^{(b)})} \right|. \tag{7.57}
\end{aligned}$$

Introducing the abbreviations

$$A(a, a) := \det_{i,j=1,\dots,2n_a} \left(\frac{1}{w_i^{(a)} - z_j^{(a)}} \right), \tag{7.58}$$

and

$$B(a, b) := \det_{i,j=1,\dots,2(n_a+n_b)} \left(\frac{1}{w_i^{(a,b)} - z_j^{(a,b)}} \right), \tag{7.59}$$

one can write \mathcal{F} in the handy form (compare (7.53))

$$\begin{aligned}
\mathcal{F} &= \left[\prod_{a=1}^N |A(a, a)| \right]^{1 - \frac{g}{\pi+gN} - (N-1)\frac{g}{\pi+gN}} \left[\prod_{a < b}^N |B(a, b)| \right]^{\frac{g}{\pi+gN}} \\
&= \left[\prod_{a=1}^N |A(a, a)| \right]^{\frac{\pi}{\pi+gN}} \left[\prod_{a < b}^N |B(a, b)| \right]^{\frac{g}{\pi+gN}}, \tag{7.60}
\end{aligned}$$

where I used

$$|B(a, b)| = |B(b, a)|, \tag{7.61}$$

which can be seen to hold from (7.54) and (7.59).

Step 11 : Hölder's inequality

Let

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{7.62}$$

Then the following chain of inequalities holds

$$\begin{aligned}
& \int_{\Lambda} \prod_{b=1}^N \prod_{i=1}^{n_b} d^2 x_i^{(b)} d^2 y_i^{(b)} \mathcal{F}(\{x, y\}) \\
& \leq \left[\int_{\Lambda} \prod_{a=1}^N \prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \prod_{b=1}^N |A(b, b)|^{\frac{\pi}{\pi+gN}p} \right]^{\frac{1}{p}} \\
& \times \left[\int_{\Lambda} \prod_{c=1}^N \left[\prod_{j=1}^{n_c} d^2 x_j^{(c)} d^2 y_j^{(c)} \right] \prod_{d < e}^N |B(d, e)|^{\frac{g}{\pi+gN}q} \right]^{\frac{1}{q}} \\
& \leq \left[\int_{\Lambda} \prod_{a=1}^N \left[\prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \right] \prod_{a=1}^N |A(a, a)|^{\frac{\pi}{\pi+gN}p} \right]^{\frac{1}{p}} \\
& \times \prod_{a < b}^N \left[\int_{\Lambda} \left[\prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \prod_{j=1}^{n_b} d^2 x_j^{(b)} d^2 y_j^{(b)} \right] |B(a, b)|^{\frac{g}{\pi+gN}q(N-1)} \right]^{\frac{1}{q(N-1)}}. \quad (7.63)
\end{aligned}$$

In the first step the usual Hölder inequality was used, and in the second step I applied Corollary B.2 (Equations (B.48) - (B.51)) proven in Appendix B.7.

To apply the bounds on the integrals over the determinants obtained in Appendix B.8, the exponents in (7.63) have to obey the following inequalities

$$\frac{\pi}{\pi + gN} p < 1, \quad (7.64)$$

and

$$\frac{g}{\pi + gN} q (N - 1) < 1. \quad (7.65)$$

Using (7.62) both restrictions can be rewritten in terms of q leading to

$$\frac{1}{q} \in \frac{g}{\pi + gN} ((N - 1), N). \quad (7.66)$$

Since the interval on the right hand side of (7.66) is not empty, such a q can be found and fixes p via (7.62). (B.57) then implies

$$\int_{\Lambda} \left[\prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \right] |A(a, a)|^{\frac{\pi}{\pi+gN}p} \leq (2n_a)! \left[\Xi \left(\frac{\pi}{\pi + gN} p \right) \right]^{2n_a}, \quad (7.67)$$

and

$$\begin{aligned}
& \int_{\Lambda} \left[\prod_{i=1}^{n_a} d^2 x_i^{(a)} d^2 y_i^{(a)} \prod_{j=1}^{n_b} d^2 x_j^{(b)} d^2 y_j^{(b)} \right] |B(a, b)|^{\frac{g}{\pi+gN}q(N-1)} \\
& \leq (2n_a + 2n_b)! \left[\Xi \left(\frac{g}{\pi + gN} q (N - 1) \right) \right]^{2(n_a + n_b)}. \quad (7.68)
\end{aligned}$$

The constants $\Xi(\cdot)$ do not depend on the numbers of arguments $2n_a$, $2(n_a + n_b)$. The bounds (7.67) and (7.68) establish

$$\left| A(a, a) \right|^{\frac{\pi}{\pi + gN}} \in L^p(\mathbb{R}^{4n_a}) \quad \text{and} \quad \left| B(a, b) \right|^{\frac{g}{\pi + gN}q} \in L^{N-1}(\mathbb{R}^{4(n_a + n_b)}), \quad (7.69)$$

which are necessary conditions for the application of Hölder's inequality and the Corollary B.2 in (7.63). Define

$$\xi := \max \left\{ \Xi\left(\frac{\pi}{\pi + gN}p\right), \Xi\left(\frac{g}{\pi + gN}q(N-1)\right) \right\}. \quad (7.70)$$

Thus

$$\begin{aligned} & \int_{\Lambda} \prod_{b=1}^N \prod_{i=1}^{n_b} d^2 x_i^{(b)} d^2 y_i^{(b)} \mathcal{F}(\{x, y\}) \\ & \leq \left[\prod_{a=1}^N (2n_a)! \right]^{\frac{1}{p}} \left[\prod_{b < c}^N (2n_b + 2n_c)! \right]^{\frac{1}{q(N-1)}} \prod_{d=1}^N (\xi)^{\frac{1}{p}2n_d} \prod_{e < f}^N (\xi)^{\frac{1}{q(N-1)}(2n_e + 2n_f)} \\ & = \left[\prod_{a=1}^N (2n_a)! \right]^{\frac{1}{p}} \left[\prod_{b < c}^N (2n_b + 2n_c)! \right]^{\frac{1}{q(N-1)}} \prod_{d=1}^N (\xi)^{2n_d}, \end{aligned} \quad (7.71)$$

where I used

$$\sum_{a < b}^N (n_a + n_b) = (N-1) \sum_{a=1}^N n_a, \quad (7.72)$$

and (7.62) in the last step. Finally (see (7.52))

$$\begin{aligned} & \prod_{a=1}^N \frac{1}{(2n_a)!} \left\langle \prod_{b=1}^N [\chi_+^{(b)} [Q^{0,S}] \chi_-^{(b)} [Q^{0,S}]]^{n_b} \right\rangle_{Q^{0,S}} \\ & \leq \prod_{a=1}^N (\xi 8^{[1 - \frac{g}{\pi + gN}]})^{2n_a} \prod_{b=1}^N [(2n_b)!]^{\frac{1}{p}-1} \prod_{c < d}^N [(2n_c + 2n_d)!]^{\frac{1}{q(N-1)}}. \end{aligned} \quad (7.73)$$

Using

$$\begin{aligned} & \prod_{a=1}^N [(2n_a)!]^{\frac{1}{p}-1} \prod_{b < c}^N [(2n_b + 2n_c)!]^{\frac{1}{q(N-1)}} = \prod_{a=1}^N [(2n_a)!]^{\frac{-1}{q}} \prod_{b < c}^N [(2n_b + 2n_c)!]^{\frac{1}{q(N-1)}} \\ & = \left[\prod_{a < b}^N \frac{(2n_a + 2n_b)!}{(2n_a)!(2n_b)!} \right]^{\frac{1}{q(N-1)}} = \left[\prod_{a < b}^N \binom{2n_a + 2n_b}{2n_a} \right]^{\frac{1}{q(N-1)}} \\ & \leq \left[\prod_{a < b}^N 2^{(2n_a + 2n_b)} \right]^{\frac{1}{q(N-1)}} = \prod_{a=1}^N 2^{2n_a \frac{1}{q}} \end{aligned} \quad (7.74)$$

one concludes

$$\prod_{a=1}^N \frac{1}{(2n_a)!} \left\langle \prod_{b=1}^N [\chi_+^{(b)} [Q^{0,S}] \chi_-^{(b)} [Q^{0,S}]]^{n_b} \right\rangle_{Q^{0,S}} \leq \prod_{a=1}^N \left(\xi 8^{[1-\frac{g}{\pi+gN}]} 2^{\frac{1}{q}} \right)^{2n_a} . \quad (7.75)$$

Inserting this into the series (7.47) the final bound is obtained as

$$\mathcal{E} \leq 2^N \prod_{a=1}^N \sum_{n_a=0}^{\infty} \left(\frac{\beta^{(a)}}{r} \right)^{2n_a} , \quad (7.76)$$

with

$$r := \left[2\omega \xi 8^{[1-\frac{g}{\pi+gN}]} 2^{\frac{1}{q}} \right]^{-1} . \quad (7.77)$$

Clearly the series converges if

$$\beta^{(a)} < r , \quad \forall a = 1, \dots, N , \quad (7.78)$$

and (7.10) then holds. \square

7.3 Remarks on the mass perturbation

The convergence of the mass perturbation series in the presence of the space-time cutoff Λ is a nice result. In particular it establishes the existence of the model for small quark masses and finite Λ . But in order to extract the physical spectrum one has to send Λ to infinity since it breaks translation invariance. For the $N = 1$ flavor case Fröhlich and Seiler [27] using the Cluster Expansion, were able to remove Λ nonperturbatively. Below it will be shown for the first few terms of the expansion, that for $N = 1$ it is even possible to send $\Lambda \rightarrow \infty$ termwise. For $N > 1$ it turns out that the known methods to remove Λ do not work. Of course it should be possible to remove Λ nonperturbatively. This would either require an adaption of the cluster expansion, or even some new techniques as will be argued in the end of this section.

Below I will discuss the problem with taking the termwise limit, as it shows up when one tries to compute the masses of the particles that correspond to the Cartan currents. In order to extract the self energies one has to compute the fully connected two point functions of the Cartan currents. The generating functional $W[\alpha]$ for connected correlation functions is given by

$$W[\alpha] := \ln \left(Z[\alpha] \right) , \quad (7.79)$$

where

$$Z[\alpha] := \lim_{\mu \rightarrow 0} \left\langle e^{i \sum_{I=1}^N (\Phi^{(I)}, \alpha^{(I)})} e^{-S_{int}} \right\rangle_{K^\mu} . \quad (7.80)$$

The interaction term S_{int} is given by (see (7.3))

$$S_{int}[\Phi^{(I)}] := -\frac{1}{2\pi} \sum_{a=1}^N m^{(a)} c^{(a)} \int_{\Lambda} d^2x$$

$$\left[\prod_{I=1}^N : e^{-i2\sqrt{\pi}\omega(I)U_{Ia}\Phi^{(I)}(x)} :_{M^{(I)}} e^{+i\frac{\theta}{N}} + \prod_{I=1}^N : e^{+i2\sqrt{\pi}\omega(I)U_{Ia}\Phi^{(I)}(x)} :_{M^{(I)}} e^{-i\frac{\theta}{N}} \right]. \quad (7.81)$$

I introduced

$$\omega^{(1)} := \sqrt{\frac{\pi}{\pi + gN}} \quad , \quad \omega^{(I)} := 1 \quad \text{for } 2 \leq I \leq N, \quad (7.82)$$

to account for the factors in the Wick ordered exponential properly (compare (7.2)). The test function $t(x)$ that shows up in (7.2) was replaced by the characteristic function of the finite rectangle Λ in space time. The covariance K^μ reads (compare (6.35) and (6.36))

$$K^\mu := \text{diag} \left(\frac{1}{-\Delta + m_d^2}, \frac{1}{-\Delta + \mu}, \dots, \frac{1}{-\Delta + \mu} \right), \quad (7.83)$$

where I introduced

$$m_d := e \sqrt{\frac{N}{\pi + gN}}, \quad (7.84)$$

for the dynamically generated mass. The Wick ordering masses $M^{(I)}$ that enter S_{int} are fixed to

$$M^{(1)} := m_d \quad , \quad M^{(I)} := 1 \quad \text{for } 2 \leq I \leq N. \quad (7.85)$$

Finally I remark that the sources $\alpha^{(I)}$ in (7.80) have to be chosen neutral

$$\int d^2x \alpha^{(I)}(x) = 0 \quad \text{for } 1 \leq I \leq N. \quad (7.86)$$

Expansion of $Z[\alpha]$ up to third order in the fermion masses gives

$$Z[\alpha] = Z^{(0)}[\alpha] + Z^{(1)}[\alpha] + \frac{1}{2}Z^{(2)}[\alpha] + O(m^3), \quad (7.87)$$

where

$$Z^{(0)}[\alpha] := \lim_{\mu \rightarrow 0} \left\langle e^{+i \sum_{I=1}^N (\Phi^{(I)}, \alpha^{(I)})} \right\rangle_{K^\mu}, \quad (7.88)$$

and

$$Z^{(1)}[\alpha] := \frac{1}{2\pi} \sum_{a=1}^N m^{(a)} c^{(a)} \int_{\Lambda} d^2x \lim_{\mu \rightarrow 0} \left\langle e^{+i \sum_{I=1}^N (\Phi^{(I)}, \alpha^{(I)})} \right\rangle$$

$$\left[\prod_{I=1}^N : e^{-i2\sqrt{\pi}\omega(I)U_{Ia}\Phi^{(I)}(x)} :_{M^{(I)}} e^{+i\frac{\theta}{N}} + \prod_{I=1}^N : e^{+i2\sqrt{\pi}\omega(I)U_{Ia}\Phi^{(I)}(x)} :_{M^{(I)}} e^{-i\frac{\theta}{N}} \right] \Bigg\rangle_{K^\mu}, \quad (7.89)$$

and finally

$$Z^{(2)}[\alpha] := \frac{1}{(2\pi)^2} \sum_{a_1, a_2=1}^N \left[\prod_{i=1}^2 m^{(a_i)} c^{(a_i)} \int_{\Lambda} d^2 x_i \right] \lim_{\mu \rightarrow 0} \left\langle e^{+i \sum_{I=1}^N (\Phi^{(I)}, \alpha^{(I)})} \prod_{j=1}^2 \left[\prod_{I=1}^N : e^{-i2\sqrt{\pi}\omega(I)U_{Ia_j}\Phi^{(I)}(x_j)} :_{M^{(I)}} e^{+i\frac{\theta}{N}} + \prod_{I=1}^N : e^{+i2\sqrt{\pi}\omega(I)U_{Ia_j}\Phi^{(I)}(x_j)} :_{M^{(I)}} e^{-i\frac{\theta}{N}} \right] \right\rangle_{K^\mu}. \quad (7.90)$$

$Z^{(0)}$ can be evaluated easily (see (A.21) and (A.38)) and gives

$$Z^{(0)} = e^{-\frac{1}{2}(\alpha^{(1)}, C_{m_d} \alpha^{(1)})} \prod_{I=2}^N e^{-\frac{1}{2}(\alpha^{(I)}, C_0 \alpha^{(I)})}, \quad (7.91)$$

where the massive covariance C_{m_d} is given by (A.33) and the massless covariance C_0 by (A.40).

The expectation values that show up in $Z^{(1)}$ factorize with respect to the flavors, and give rise to

$$\begin{aligned} & \left\langle e^{+i(\Phi^{(1)}, \alpha^{(1)})} : e^{\pm i2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+gN}}U_{1a}\Phi^{(1)}(x)} :_{m_d} \right\rangle_{C_{m_d}} \\ & \times \prod_{I=2}^N \lim_{\mu \rightarrow 0} \left\langle e^{+i(\Phi^{(I)}, \alpha^{(I)})} : e^{\pm i2\sqrt{\pi}U_{Ia}\Phi^{(I)}(x)} :_1 \right\rangle_{C_\mu}. \end{aligned} \quad (7.92)$$

For $N \geq 2$ these terms vanish. The reason for this is that for $I = 2$ the neutrality condition is never fulfilled. In particular, since α was chosen neutral (see (7.86)), U_{2a} would have to vanish for neutrality. But U_{2a} can only assume the values $1/\sqrt{(N-1+(N-1)^2)}$ and $-(N-1)/\sqrt{(N-1+(N-1)^2)}$ as can be seen from (B.17). Hence neutrality is violated and the expectation values (7.92) are all equal to zero and thus $Z^{(1)}$ vanishes. The situation is different for the $N = 1$ case, since no massless particles are involved then. The result for $Z_1^{(1)}$ (to distinguish the $N = 1$ flavor result from the general $Z^{(1)}$ an extra subscript 1 was added) is given by

$$Z_1^{(1)}[\alpha] = Z_1^{(0)}[\alpha] \frac{1}{2\pi} mc \int_{\Lambda} d^2 x \left[e^{+2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x), C_{m_d} \alpha) + i\theta} + e^{-2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x), C_{m_d} \alpha) - i\theta} \right]. \quad (7.93)$$

All flavor indices are suppressed. The convolution with the δ -functional is understood as $(\delta(x), t) = t(x)$.

The expectation values that enter $Z^{(2)}$ also factorize and are given by

$$\begin{aligned}
& e^{+i2\frac{\theta}{N}} \lim_{\mu \rightarrow 0} \prod_{I=1}^N \left\langle e^{i(\Phi^{(I)}, \alpha^{(I)})} : e^{-i2\sqrt{\pi}\omega^{(I)} U_{Ia} \Phi^{(I)}(x)} :_{M^{(I)}} : e^{-i2\sqrt{\pi}\omega^{(I)} U_{Ib} \Phi^{(I)}(y)} :_{M^{(I)}} \right\rangle_{K_{II}^\mu} \\
& + \lim_{\mu \rightarrow 0} \prod_{I=1}^N \left\langle e^{i(\Phi^{(I)}, \alpha^{(I)})} : e^{-i2\sqrt{\pi}\omega^{(I)} U_{Ia} \Phi^{(I)}(x)} :_{M^{(I)}} : e^{+i2\sqrt{\pi}\omega^{(I)} U_{Ib} \Phi^{(I)}(y)} :_{M^{(I)}} \right\rangle_{K_{II}^\mu} \\
& + \lim_{\mu \rightarrow 0} \prod_{I=1}^N \left\langle e^{i(\Phi^{(I)}, \alpha^{(I)})} : e^{+i2\sqrt{\pi}\omega^{(I)} U_{Ia} \Phi^{(I)}(x)} :_{M^{(I)}} : e^{-i2\sqrt{\pi}\omega^{(I)} U_{Ib} \Phi^{(I)}(y)} :_{M^{(I)}} \right\rangle_{K_{II}^\mu} \\
& + e^{-i2\frac{\theta}{N}} \lim_{\mu \rightarrow 0} \prod_{I=1}^N \left\langle e^{i(\Phi^{(I)}, \alpha^{(I)})} : e^{+i2\sqrt{\pi}\omega^{(I)} U_{Ia} \Phi^{(I)}(x)} :_{M^{(I)}} : e^{+i2\sqrt{\pi}\omega^{(I)} U_{Ib} \Phi^{(I)}(y)} :_{M^{(I)}} \right\rangle_{K_{II}^\mu} .
\end{aligned} \tag{7.94}$$

Since the $\alpha^{(I)}$ are neutral, the neutrality is again determined by the U_{Ia} , $2 \leq I \leq N$. For the first and the last term in (7.94) the condition for nonvanishing contributions reads

$$U_{Ia} + U_{Ib} \stackrel{!}{=} 0 \quad \forall I = 2, \dots, N. \tag{7.95}$$

Define N vectors $\vec{h}^{(a)}$ $a = 1, 2, \dots, N$ each with $N - 1$ entries

$$\vec{h}_I^{(a)} := U_{Ia}, \quad I = 2, 3, \dots, n. \tag{7.96}$$

Thus the neutrality condition reads

$$\vec{h}^{(a)} + \vec{h}^{(b)} \stackrel{!}{=} 0. \tag{7.97}$$

Inspecting (B.17) shows that for $N > 2$ none of the vectors $\vec{h}^{(a)}$ is the negative of another one, and (7.97) can never be fulfilled then. For the following I restrict myself to the case $N \geq 3$ and hence the first and the last terms in (7.94) vanish.

The neutrality condition for the second and the third term in (7.94) reads

$$\vec{h}^{(a)} - \vec{h}^{(b)} \stackrel{!}{=} 0. \tag{7.98}$$

Inspecting (B.17) again shows that for none of the vectors $\vec{h}^{(a)}$ is equal to any other, and (7.98) only has the trivial solution $a = b$. Thus for $N \geq 3$ one obtains

$$\begin{aligned}
Z^{(2)}[\alpha] &= Z^{(0)}[\alpha] \frac{2}{(2\pi)^2} \sum_{a=1}^N \left(m^{(a)} c^{(a)} \right)^2 \int_{\Lambda} d^2x d^2y \\
& e^{+2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1a} (\delta(x)-\delta(y), C_{m_d} \alpha^{(1)})} \prod_{I=2}^N e^{+2\sqrt{\pi} U_{Ia} (\delta(x)-\delta(y), C_0 \alpha^{(I)})} \rho(x-y), \tag{7.99}
\end{aligned}$$

where

$$\begin{aligned}
\rho(x-y) &:= \lim_{\mu \rightarrow 0} \prod_{I=1}^N \left\langle : e^{-i2\sqrt{\pi}\omega^{(I)}U_{Ia}\Phi^{(I)}(x)} :_{M^{(I)}} : e^{+i2\sqrt{\pi}\omega^{(I)}U_{Ia}\Phi^{(I)}(y)} :_{M^{(I)}} \right\rangle_{K_{II}^\mu} \\
&= e^{4\pi\frac{\pi}{\pi+gN}\frac{1}{N}C_{m_d}(x-y)} \prod_{I=2}^N e^{4\pi(U_{Ia})^2 C_0(x-y)} \\
&= e^{\frac{2}{N}\frac{\pi}{\pi+gN}K_0(m_d|x-y|)} \left(\frac{1}{(x-y)^2} \right)^{\frac{N-1}{N}} \left(\frac{e^{2\gamma}}{4} \right)^{-\frac{N-1}{N}}. \tag{7.100}
\end{aligned}$$

In the last step I inserted (A.33) and (A.40) for the covariances and used (B.20) to remove the U_{Ia} , which shows that ρ does not depend on a .

Again the result for $N = 1$ flavor is different, and I quote it for later reference

$$\begin{aligned}
Z_1^{(2)}[\alpha] &= Z_1^{(0)}[\alpha] \frac{1}{(2\pi)^2} (mc)^2 \int_{\Lambda} d^2x d^2y \\
&\left[2e^{+2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x)-\delta(y), C_{m_d}\alpha)} + e^{+2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x)+\delta(y), C_{m_d}\alpha)} e^{i2\theta} \right. \\
&\left. + e^{-2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x)+\delta(y), C_{m_d}\alpha)} e^{-i2\theta} \right] \rho_1(x-y), \tag{7.101}
\end{aligned}$$

and

$$\rho_1(x-y) := e^{2\frac{\pi}{\pi+g}K_0(m_d|x-y|)}. \tag{7.102}$$

Putting things together one obtains the general structure

$$Z[\alpha] = Z^{(0)}[\alpha] \left[1 + \tilde{Z}^{(1)}[\alpha] + \frac{1}{2}\tilde{Z}^{(2)}[\alpha] + O(m^3) \right], \tag{7.103}$$

which is correct for all N , since $Z^{(0)}$ always factorizes. The $\tilde{Z}^{(I)}[\alpha]$ can be read off from (7.99) ((7.93), (7.101) for $N = 1$). Thus the expansion of $W[\alpha]$ up to third order in the fermion masses reads

$$W[\alpha] = \ln(Z^{(0)}[\alpha]) + \tilde{Z}^{(1)}[\alpha] + \frac{1}{2}\tilde{Z}^{(2)}[\alpha] - \frac{1}{2}(\tilde{Z}^{(1)}[\alpha])^2 + O(m^3). \tag{7.104}$$

Before the more involved case of more than one flavors will be attacked, I discuss $N = 1$. The generating functional then reads

$$\begin{aligned}
W[\alpha] &= -\frac{1}{2}(\alpha, C_{m_d}\alpha) \\
&+ \frac{1}{2\pi}mc \int_{\Lambda} d^2x \left[e^{+2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x), C_{m_d}\alpha)+i\theta} + e^{-2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x), C_{m_d}\alpha)-i\theta} \right] \\
&+ \frac{1}{2} \frac{1}{(2\pi)^2} (mc)^2 \int_{\Lambda} d^2x d^2y \left[2e^{+2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x)-\delta(y), C_{m_d}\alpha)} \right.
\end{aligned}$$

$$\begin{aligned}
& + e^{+2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x)+\delta(y), C_{m_d}\alpha)} e^{i2\theta} + e^{-2\sqrt{\pi}\sqrt{\frac{\pi}{\pi+g}}(\delta(x)+\delta(y), C_{m_d}\alpha)} e^{-i2\theta} \Big] \\
& \times [\rho_1(x-y) - 1] + O(m^3) . \tag{7.105}
\end{aligned}$$

The connected two point function is given by

$$\begin{aligned}
G(w, z) &:= \langle \Phi(w)\Phi(z) \rangle_c = - \frac{\delta^2}{\delta\alpha(w) \delta\alpha(z)} W[\alpha] \Big|_{\alpha=0} \\
&= C_{m_d}(w-z) - \frac{mc}{2\pi} \frac{4\pi^2}{\pi+g} \int_{\Lambda} d^2x C_{m_d}(x-w) C_{m_d}(x-z) 2 \cos(\theta) \\
&- \frac{1}{2} \left(\frac{mc}{2\pi} \right)^2 \frac{4\pi^2}{\pi+g} \int_{\Lambda} d^2x d^2y \left\{ 2 [C_{m_d}(x-w) - C_{m_d}(y-w)] [C_{m_d}(x-z) - C_{m_d}(y-z)] \right. \\
&\quad \left. + 2 \cos(2\theta) [C_{m_d}(x-w) + C_{m_d}(y-w)] [C_{m_d}(x-z) + C_{m_d}(y-z)] \right\} \\
&\times [\rho_1(x-y) - 1] + O(m^3) . \tag{7.106}
\end{aligned}$$

To obtain the Fourier transform of the propagator I insert

$$C_{m_d}(\xi) = \frac{1}{2\pi} \int d^2p \hat{C}_{m_d}(p) e^{ip\xi} = \frac{1}{(2\pi)^2} \int d^2p \frac{1}{p^2 + m_d^2} e^{ip\xi} , \tag{7.107}$$

and end up with

$$\begin{aligned}
G(w, z) &= \frac{1}{2\pi} \int d^2p \hat{C}_{m_d}(p) e^{ip(w-z)} \\
&- \frac{mc}{2\pi} \frac{4\pi^2}{\pi+g} \frac{1}{(2\pi)^2} \int d^2p d^2q \hat{C}_{m_d}(p) \hat{C}_{m_d}(q) \int_{\Lambda} d^2x e^{ip(x-w)} e^{iq(x-z)} 2 \cos(\theta) \\
&- \frac{1}{2} \left(\frac{mc}{2\pi} \right)^2 \frac{4\pi^2}{\pi+g} \frac{1}{(2\pi)^2} \int d^2p d^2q \hat{C}_{m_d}(p) \hat{C}_{m_d}(q) \int_{\Lambda'} d^2x d^2\xi e^{ix(p+q)} \\
&\times \left\{ 2[1 - e^{-ip\xi} - e^{-iq\xi} + e^{i\xi(p+q)}] + 2 \cos(2\theta)[1 + e^{-ip\xi} + e^{-iq\xi} + e^{i\xi(p+q)}] \right\} \\
&\times [\rho_1(\xi) - 1] + O(m^3) . \tag{7.108}
\end{aligned}$$

The transformation $x - y := \xi$ of the arguments was performed in the last step, which changes the rectangle Λ into some other finite area Λ' . Inserting the expansion (A.6) in (7.102), one finds

$$\rho_1(\xi) \sim \left(\frac{1}{(\xi)^2} \right)^{\frac{\pi}{\pi+g}} \quad \text{for } \xi \rightarrow 0 , \tag{7.109}$$

which shows that the short distance singularity of ρ_1 is integrable for $g > 0$. Furthermore since K_0 approaches zero exponentially, the Fourier transform of

$$\rho_1(\xi) - 1 , \tag{7.110}$$

exists as can be seen from (7.102). This allows to send the cutoff to infinity. Using (B.4) one obtains for the propagator in momentum space

$$\begin{aligned} \hat{G}(p) &= \hat{C}_{m_d}(p) - mc \frac{4\pi^2}{\pi+g} \left(\hat{C}_{m_d}(p) \right)^2 2 \cos(\theta) - (mc)^2 \frac{4\pi^2}{\pi+g} \left(\hat{C}_{m_d}(p) \right)^2 \\ &\times \left\{ [2 + 2 \cos(2\theta)](\widehat{\rho_1 - 1})(0) - [2 - 2 \cos(2\theta)](\widehat{\rho_1 - 1})(p) \right\} + O(m^3) . \end{aligned} \quad (7.111)$$

$\hat{G}(p)$ can now be inverted easily and the self energy can be computed.

The situation for $N > 1$ is different, since the term $Z^{(1)}$ linear in the fermion masses vanishes due to the neutrality condition (see (7.92)). Inserting (7.91) and (7.99) into (7.104) gives

$$\begin{aligned} W[\alpha] &= -\frac{1}{2} \left(\alpha^{(1)}, C_{m_d} \alpha^{(1)} \right) - \frac{1}{2} \sum_{I=2}^N \left(\alpha^{(I)}, C_0 \alpha^{(I)} \right) \\ &+ \frac{1}{(2\pi)^2} \sum_{a=1}^N \left(m^{(a)} c^{(a)} \right)^2 \int_{\Lambda} d^2x d^2y \\ &e^{+2\sqrt{\pi} \sqrt{\frac{\pi}{\pi+gN}} U_{1a}(\delta(x)-\delta(y), C_{m_d} \alpha^{(1)})} \prod_{I=2}^N e^{+2\sqrt{\pi} U_{Ia}(\delta(x)-\delta(y), C_0 \alpha^{(I)})} \rho(x-y) + O(m^3) . \end{aligned} \quad (7.112)$$

The connected two point function for $\Phi^{(1)}$ is given by

$$\begin{aligned} G^{(1)}(w-z) &:= \left\langle \Phi^{(1)}(w) \Phi^{(1)}(z) \right\rangle_c = C_{m_d}(w-z) \\ &- \frac{1}{\pi+gN} \frac{1}{N} \sum_{a=1}^N \left(m^{(a)} c^{(a)} \right)^2 \int_{\Lambda} d^2x d^2y \\ &\times \left[C_{m_d}(x-z) - C_{m_d}(y-z) \right] \left[C_{m_d}(x-w) - C_{m_d}(y-w) \right] \rho(x-y) + O(m^3) . \end{aligned} \quad (7.113)$$

Inserting (7.107) one obtains

$$\begin{aligned} G^{(1)}(w-z) &= \frac{1}{2\pi} \int d^2p \hat{C}_{m_d}(p) e^{ip(w-z)} \\ &- \frac{1}{\pi+gN} \frac{1}{N} \sum_{a=1}^N \left(m^{(a)} c^{(a)} \right)^2 \frac{1}{(2\pi)^2} \int d^2p d^2q \hat{C}_{m_d}(p) \hat{C}_{m_d}(q) \int_{\Lambda'} d^2x d^2\xi e^{ix(p+q)} \\ &2[1 - e^{-ip\xi} - e^{-iq\xi} + e^{i\xi(p+q)}] \rho(\xi) + O(m^3) . \end{aligned} \quad (7.114)$$

The properties of ρ have to be discussed. Inserting the expansion (A.6) one obtains from (7.100)

$$\rho(\xi) \sim \left(\frac{1}{\xi^2} \right)^{1 - \frac{1}{N} \left(1 - \frac{\pi}{\pi+gN} \right)} \quad \text{for } \xi \rightarrow 0 . \quad (7.115)$$

This shows that the short distance singularity is integrable also for $N > 1$. Unfortunately there remains an infrared problem. Since K_0 approaches 0 for large argument, ρ behaves as

$$\rho(\xi) \sim \left(\frac{1}{\xi^2}\right)^{1-\frac{1}{N}} \quad \text{for } \xi \rightarrow \infty, \quad (7.116)$$

as can be seen from (7.100). This implies that the cutoff cannot be removed properly in the expression for $G^{(1)}$. The analysis of the propagators for $\Phi^{(I)}$, $I > 2$ can be taken over by replacing $C_{m_d} \rightarrow C_0$. The infrared problem remains, and one has to conclude that the propagators in momentum space, and thus the self energies cannot be computed termwise.

The reason why the cutoff Λ cannot be removed termwise is related to the fact that the fermion determinant in infinite volume behaves as $m^2 \ln(m)$ for small mass m (compare Sections 4.1, 4.2). Thus the power series expansion of the determinant is only correct for finite cutoff Λ . If one could somehow reorder or sum up the expansion to extract the $m^2 \ln(m)$ behaviour it might be possible to send Λ to infinity also for $N > 1$.

7.4 Semiclassical approximation

Since the extraction of physical results from the mass perturbation with the Λ -cutoff present is rather problematic, one can try to learn something from a semiclassical approximation of the Lagrangian

$$\begin{aligned} \mathcal{L}_{GSG} = & \frac{1}{2} \sum_{I=1}^N \partial_\mu \Phi^{(I)} \partial_\mu \Phi^{(I)} + \frac{1}{2} (\Phi^{(1)})^2 \frac{e^2 N}{\pi + gN} \\ & - \frac{1}{\pi} \sum_{b=1}^N m^{(b)} c^{(b)} \cos \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi + gN}} U_{1b} \Phi^{(1)} + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi^{(I)} - \frac{\theta}{N} \right). \end{aligned} \quad (7.117)$$

The coefficients $c^{(b)}$ are given by (c.f. (6.51))

$$c^{(b)} = \left(\frac{M^{(1)} e^\gamma}{2} \right)^{\frac{\pi}{\pi + gN} \frac{1}{N}} \prod_{I=2}^N \left(\frac{M^{(I)} e^\gamma}{2} \right)^{(U_{Ib})^2}. \quad (7.118)$$

The $M^{(I)}$ are the masses that are used for Wick ordering. They are still free parameters. A natural choice is to Wick order the fields with respect to their own mass. Such a restriction can be used to fix the $M^{(I)}$ in the end. To simplify the rather involved structure I consider a special case defined by

1:

$$m^{(b)} = m \quad \forall b = 1, 2 \dots N. \quad (7.119)$$

2:

$$M^{(I)} = M \quad \forall I = 2, 3 \dots N . \quad (7.120)$$

The first condition assumes equal masses for all fermion fields. The fields $\Phi^{(I)}$, $I > 1$ are treated symmetrically by the Lagrangian then. The second restriction thus makes use of the fact that only $\Phi^{(1)}$ plays an extra role in \mathcal{L}_{GSG} . Hence it makes sense to Wick order those fields with respect to the same mass which is expressed by (7.120). This implies for the constants $c^{(b)}$

$$c^{(b)} = \left(\frac{M^{(1)} e^\gamma}{2} \right)^{\frac{\pi}{\pi+gN} \frac{1}{N}} \left(\frac{M e^\gamma}{2} \right)^{1-\frac{1}{N}} =: c \quad \forall b = 1, 2 \dots N , \quad (7.121)$$

where I made use of (B.20) to remove the U_{Ib} and thus the dependence on b .

Thus the simplified Lagrangian reads

$$\mathcal{L}_{GSG} = \frac{1}{2} \sum_{I=1}^N \partial_\mu \Phi^{(I)} \partial_\mu \Phi^{(I)} + V(\Phi^{(I)}) , \quad (7.122)$$

where the potential $V(\Phi^{(I)})$ is given by

$$\frac{1}{2} (\Phi^{(1)})^2 \frac{e^2 N}{\pi + gN} - \frac{1}{\pi} mc \sum_{b=1}^N \cos \left(2\sqrt{\frac{\pi}{N}} \sqrt{\frac{\pi}{\pi+gN}} \Phi^{(1)} + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi^{(I)} - \frac{\theta}{N} \right) . \quad (7.123)$$

It is rather instructive to plot the potential in the $N = 2$ case. In order to point out the interesting features of the potential, I have chosen the numerical values $g = 0$, $e^2 2/(\pi + g2) = 1$, $mc/\pi = 0.8$, $\theta/2 = 0$ for Figure 7.1 (next page).

Obviously there are infinitely many degenerate minima. For the semiclassical approximation one has to find those minima $\Phi_0^{(I)}$ of $V(\Phi^{(I)})$, i.e. one has to solve the equations

$$0 \stackrel{!}{=} \left. \frac{\partial}{\partial \Phi^{(1)}} V(\Phi^{(I)}) \right|_{\Phi^{(I)} = \Phi_0^{(I)}} = \frac{e^2 N}{\pi + gN} \Phi_0^{(1)} + \frac{2mc}{\sqrt{\pi + gN}} \sum_{b=1}^N \frac{1}{\sqrt{N}} \sin \left(2\sqrt{\frac{\pi}{N}} \sqrt{\frac{\pi}{\pi+gN}} \Phi_0^{(1)} + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi_0^{(I)} - \frac{\theta}{N} \right) , \quad (7.124)$$

and for $J = 2, 3, \dots N$

$$0 \stackrel{!}{=} \left. \frac{\partial}{\partial \Phi^{(J)}} V(\Phi^{(I)}) \right|_{\Phi^{(I)} = \Phi_0^{(I)}} = \frac{2mc}{\sqrt{\pi}} \sum_{b=1}^N U_{Jb} \sin \left(2\sqrt{\frac{\pi}{N}} \sqrt{\frac{\pi}{\pi+gN}} \Phi_0^{(1)} + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi_0^{(I)} - \frac{\theta}{N} \right) . \quad (7.125)$$

Figure 7.1 : Plot of the potential $V(\Phi^{(I)})$ defined in Equation (7.123) for $N = 2$ flavors. The values of the parameters are given in the text.

Again one can interpret the lines of U (fixed J in U_{Jb}) as vectors $\vec{r}^{(J)}$ (see (B.17)) and denote Equations (7.125) as products of two vectors

$$\vec{r}^{(J)} \cdot \vec{s} \stackrel{!}{=} 0 \quad \forall J = 2, 3 \dots N, \quad (7.126)$$

where the entries of the vector \vec{s} are given by

$$s_b := \frac{2mc}{\sqrt{\pi}} \sin \left(2\sqrt{\pi} \sqrt{\frac{\pi}{\pi + gN}} \frac{1}{\sqrt{N}} \Phi_0^{(1)} + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi_0^{(I)} - \frac{\theta}{N} \right). \quad (7.127)$$

Equation (7.126) has the only solution (see Appendix B.3)

$$\vec{s} = \lambda (1, 1, \dots, 1) \quad \lambda \in \mathbb{R}. \quad (7.128)$$

Thus the set of equations (7.125) is equivalent to

$$2\sqrt{\pi} \sqrt{\frac{\pi}{\pi + gN}} \frac{1}{\sqrt{N}} \Phi_0^{(1)} + 2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi_0^{(I)} - \frac{\theta}{N} = \arcsin \left(\frac{\lambda \sqrt{\pi}}{2mc} \right), \quad (7.129)$$

for all $b = 1, 2, \dots, N$. Equation (7.124) can be used to express λ in terms of $\Phi_0^{(1)}$,

$$\lambda = -\sqrt{\frac{N}{\pi}} \frac{e^2}{\sqrt{\pi + gN}} \Phi_0^{(1)}. \quad (7.130)$$

Inserting this one obtains for the set of equations (7.129)

$$\sum_{I=2}^N U_{Ib} \Phi_0^{(I)} = \frac{1}{2\sqrt{\pi}} \left[\frac{\theta}{N} - \arcsin \left(\frac{e^2}{2mc} \sqrt{\frac{N}{\pi + gN}} \Phi_0^{(1)} \right) \right] - \frac{1}{\sqrt{N}} \sqrt{\frac{\pi}{\pi + gN}} \Phi_0^{(1)}, \quad (7.131)$$

for all $b = 1, 2, \dots, N$. Multiplication with U_{Jb} and summing over b gives

$$\sum_{I=2}^N \delta_{JI} \Phi_0^{(I)} = \delta_{J1} \left\{ \frac{\sqrt{N}}{2\sqrt{\pi}} \left[\frac{\theta}{N} - \arcsin \left(\frac{e^2}{2mc} \sqrt{\frac{N}{\pi + gN}} \Phi_0^{(1)} \right) \right] - \sqrt{\frac{\pi}{\pi + gN}} \Phi_0^{(1)} \right\}, \quad (7.132)$$

where I used the orthogonality of U and $\sum_b U_{Jb} = \delta_{J1} \sqrt{N}$. In the last expression the equations for the determination of the minima are decoupled and can be solved easily. For the case $2 \leq J \leq N$ one obtains the naive solution

$$\Phi_0^{(J)} = 0 \quad \forall J = 2, 3, \dots, N. \quad (7.133)$$

Of course there exists an infinite countable set of solutions since one can always shift the argument of the cosine in (7.123) by integer multiples of 2π giving rise to

$$2\sqrt{\pi} \sum_{I=2}^N U_{Ib} \Phi_0^{(I)} = n_b 2\pi, \quad n_b \in \mathbb{Z}, \quad \forall b = 1, 2, \dots, N. \quad (7.134)$$

Using the orthogonality of U , the last expression can be inverted and one ends up with

$$\Phi_0^{(I)} = \sqrt{\pi} \sum_{b=1}^N U_{Ib} n_b \quad I = 2, 3, \dots, N. \quad (7.135)$$

The $\Phi_0^{(1)}$ coordinate of the minimum has to fulfill the equation that emerges from (7.124) when inserting (7.134)

$$\frac{1}{\sqrt{N}} \sqrt{\frac{\pi}{\pi + gN}} \Phi_0^{(1)} = \frac{1}{2\sqrt{\pi}} \left[\frac{\theta}{N} - \arcsin \left(\frac{e^2}{2mc} \sqrt{\frac{N}{\pi + gN}} \Phi_0^{(1)} \right) \right]. \quad (7.136)$$

Obviously this is a trivial modification of the transcendental equation that determines the minimum in the one flavor case. The corresponding potential is

$$V^{(1)}(\Phi^{(1)}) := \frac{1}{2} (\Phi^{(1)})^2 \frac{e^2 N}{\pi + gN} - \frac{1}{\pi} mcN \cos \left(2\sqrt{\frac{\pi}{N}} \sqrt{\frac{\pi}{\pi + gN}} \Phi^{(1)} - \frac{\theta}{N} \right). \quad (7.137)$$

I plot the potential $V^{(1)}(\Phi^{(1)})$ for $N = 2$, $g = 0$, $e^2 2/\pi = 1$, $mc/\pi = 0.8$ and different values of $\theta/2$.

Figure 7.2 : Plot of the potential $V^{(1)}(\Phi^{(1)})$ defined in (7.137) for $N = 2$ flavors and $\theta/2 = 0, \theta/2 = \pi/2$ and $\theta/2 = \pi$. The values of the other parameters are given in the text.

For $\theta/N = 0 \bmod(2\pi)$ there is one unique absolute minimum and eventually (depending on the other parameters e, mc, g) several local minima. Shifting now θ/N towards $\pi \bmod(2\pi)$ the relative minima come down on one branch

of the potential leading to a degeneracy of the absolute minima for $\theta/N = \pi \bmod(2\pi)$.

Thus the discussion of the position of the minima can be summed up as follows. For $2 \leq I \leq N$ the minima are given by (7.135), whereas $\Phi_0^{(1)}$ has to be determined as solution of the transcendental equation (7.136). There is an infinite set of absolute minima due to (7.135), which gets doubled for $\theta = \pi/2$ as was shown in the discussion of $V^{(1)}(\Phi^{(1)})$. Thus the vacuum structure of the Generalized Sine Gordon model is rather different from the $N = 1$ model, since the semiclassical vacuum is always degenerate irrespective of the value of θ . Of course in the quantized theory this degeneracy vanishes, due to the Coleman theorem [19] since it is related to a symmetry.

To evaluate the mass matrix of the effective theory around the minima, the Hesse matrix

$$H_{II'} := \left. \frac{\partial^2 V(\Phi^{(J)})}{\partial \Phi^{(I)} \partial \Phi^{(I')}} \right|_{\Phi^{(J)} = \Phi_0^{(J)}}, \quad (7.138)$$

has to be computed. It can be evaluated easily

$$H = \text{diag}\left(\frac{e^2 N}{\pi + gN}, 0, \dots, 0\right) +$$

$$4mc \tilde{\lambda} \sum_{b=1}^N \begin{bmatrix} \frac{\pi}{\pi+gN} \frac{1}{N} & \sqrt{\frac{\pi}{\pi+gN}} \sqrt{\frac{1}{N}} U_{2b} & \cdot & \cdot & \cdot & \sqrt{\frac{\pi}{\pi+gN}} \sqrt{\frac{1}{N}} U_{Nb} \\ \sqrt{\frac{\pi}{\pi+gN}} \sqrt{\frac{1}{N}} U_{2b} & U_{2b} U_{2b} & \cdot & \cdot & \cdot & U_{2b} U_{Nb} \\ \sqrt{\frac{\pi}{\pi+gN}} \sqrt{\frac{1}{N}} U_{3b} & U_{3b} U_{2b} & & & & \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sqrt{\frac{\pi}{\pi+gN}} \sqrt{\frac{1}{N}} U_{Nb} & U_{Nb} U_{2b} & \cdot & \cdot & \cdot & U_{Nb} U_{Nb} \end{bmatrix} =$$

$$\text{diag}\left(\frac{e^2 N}{\pi + gN} + \frac{\pi}{\pi + gN} 4mc \tilde{\lambda}, 4mc \tilde{\lambda}, \dots, 4mc \tilde{\lambda}\right). \quad (7.139)$$

Where the orthogonality of U was used again. $\tilde{\lambda}$ is defined as (compare (7.129))

$$\tilde{\lambda} := \cos \arcsin \left(\frac{\lambda \sqrt{\pi}}{2mc} \right) = \sqrt{1 - \left(\frac{\lambda \sqrt{\pi}}{2mc} \right)^2}. \quad (7.140)$$

The Hesse matrix comes out as a positive definite diagonal matrix. The entries have to be interpreted as the squared masses of the fields $\Phi^{(I)}$ in an effective theory around the semiclassical vacua. The masses m_I are given by

$$m_1 := \sqrt{\left(\frac{e^2 N}{\pi} + 4mc \right)} \sqrt{\frac{\pi}{\pi + gN}} \quad \text{for } \Phi^{(1)}, \quad (7.141)$$

and

$$m_I := \sqrt{4mc} \quad \text{for } \Phi^{(I)}, \quad I = 2, 3, \dots, N. \quad (7.142)$$

It has to be pointed out that for vanishing fermion masses the semiclassical approximation gives the correct result, and thus is presumably rather good also for small masses.

7.5 Witten-Veneziano formula

The masses obtained in the semiclassical approximation will now be used to test Witten-Veneziano formulas. The Thirring coupling g is set to zero in the following. This is for two reasons. Firstly the Thirring term is not necessary in the semiclassical approximation. Of course one could modify the Witten-Veneziano formula to include a finite g . But the second more physical reason shows why one should not do this. Using the bosonization prescription (6.37) (set $g = 0$) the Thirring term (3.13) can be written as

$$\frac{1}{2} \frac{gN}{\pi} \int d^2x \left(\partial_\mu \Phi^{(1)}(x) \right) \left(\partial_\mu \Phi^{(1)}(x) \right). \quad (7.143)$$

This additional term causes an extra breaking of the symmetry of the scalar fields $\Phi^{(I)}$, which has to be distinguished from the breaking through the dynamically generated mass $m_1 = e\sqrt{N/\pi}$. It even can be seen how (7.143) leads to the extra factor attached to the $\Phi^{(1)}$ mass (compare (7.141)). (7.143) is an extra contribution to the kinetic term of $\Phi^{(1)}$. Normalizing the kinetic term canonically gives exactly the factor $\sqrt{\pi/(\pi + gN)}$ in (7.141).

For $g = 0$ the following Witten-Veneziano formula will be shown to hold:

$$m_1^2 - \frac{1}{N-1} \sum_{I=2}^N m_I^2 = \frac{4N}{(f_1^0)^2} P^0(0). \quad (7.144)$$

f_1^0 denotes the decay constant of the U(1)-pseudoscalar¹ particle in the model with vanishing fermion masses. Inserting the mass values (7.141) and (7.142) at $g = 0$, one finds that the left hand side of (7.144) reduces to

$$m_1^2 - \frac{1}{N-1} \sum_{I=2}^N m_I^2 = \frac{e^2 N}{\pi} \stackrel{!}{=} \frac{4N}{(f_1^0)^2} P^0(0). \quad (7.145)$$

In QED₂ the topological charge density $q(x)$ is given by (see (3.21))

$$q(x) = \frac{e}{2\pi} F_{12}(x). \quad (7.146)$$

¹ From my choice of the γ -algebra (see Appendix B.1) there follows $J_{5\mu}^{(I)} = -i\varepsilon_{\mu\nu} J_\nu^{(I)}$. Using (6.37) one obtains the bosonization prescription $J_{5\mu}^{(I)} \propto \partial_\mu \Phi^{(I)}$ for the axial vector currents. This implies that the properties (mass, decay constant) of the axial vector currents and the vector currents are determined by the same field $\Phi^{(I)}$.

The topological susceptibility is defined to be (compare (2.46))

$$\chi_{top} = \int \langle q(x)q(0) \rangle d^2x = \frac{e^2}{(2\pi)^2} \int \langle F_{12}(x)F_{12}(0) \rangle d^2x = \frac{e^2}{2\pi} \hat{G}_{FF}(0) , \quad (7.147)$$

where G_{FF} denotes the F_{12} propagator, and $\hat{G}_{FF}(0)$ its Fourier transform at zero momentum. Since

$$F_{12}(x) = \varepsilon_{\mu\nu} \partial_\mu A_\nu(x) , \quad (7.148)$$

the propagator G_{FF} thus is related to the gauge field propagator $Q_{\mu\nu}$ (see (4.45)) and is given by

$$G_{FF} = -\varepsilon_{\mu\nu} \partial_\nu Q_{\mu\rho} \varepsilon_{\rho\sigma} \partial_\sigma . \quad (7.149)$$

Inserting Q (4.45) and transforming to momentum space one obtains

$$\hat{G}_{FF}(p) = \frac{p^2}{p^2 + e^2 \frac{N}{\pi}} = 1 - \frac{e^2 \frac{N}{\pi}}{p^2 + e^2 \frac{N}{\pi}} = 1 - \int_0^\infty \frac{d\rho(\mu^2)}{p^2 + \mu^2} . \quad (7.150)$$

In the last step I made the spectral integral explicit. One nicely sees that the spectral measure

$$d\rho(\mu^2) = \delta(\mu^2 - m_d^2) d\mu^2 \quad (7.151)$$

is ‘dominated’ by the contribution of the U(1)-particle (compare (2.53)). From (7.147), (7.150) one immediately reads off the contact term $P(0)$ in the spectral decomposition of χ_{top}

$$P^0(0) = \frac{e^2}{4\pi^2} . \quad (7.152)$$

The last missing ingredient is the decay constant f_1^0 in the massless model. It is defined by

$$f_1^0 = m_d^{-2} \langle 0 | \partial_\mu J_{5\mu}^{(1)} | 1 \rangle , \quad (7.153)$$

where $| 1 \rangle$ is the state that corresponds to the U(1)-current. In QCD this would be the state $| \eta' \rangle$. Using the definition of the U(1)-current (3.12) the anomaly equation² for $g = 0$ takes the form (compare (3.19))

$$\partial_\mu J_{5\mu}^{(1)} = 2\sqrt{N}q , \quad (7.154)$$

which I insert in (7.153) to end up with

$$f_1^0 = m_d^{-2} 2\sqrt{N} \langle 0 | q | 1 \rangle = m_d^{-2} 2\sqrt{N} \langle 0 | q \frac{i}{m} F_{12} | 0 \rangle . \quad (7.155)$$

²Note that in my definition of the U(1)-current there is an extra factor $1/\sqrt{N}$ which modifies (3.19) by this factor, compared to the usual notation.

In the last step the state $|1\rangle$ was generated as $Z^{\frac{1}{2}}F_{12}|0\rangle$ with the normalization condition

$$\hat{G}_{11} \stackrel{!}{=} \frac{1}{p^2 + m_d^2} + \text{contact term} , \quad (7.156)$$

giving rise to $Z = -\frac{1}{m_d^2}$. One ends up with

$$f_1^0 = i \frac{1}{\sqrt{\pi}} . \quad (7.157)$$

Insertion of (7.152) and (7.157) in (7.145) gives an identity. This explicit computation shows that Equation (7.144) holds in the semiclassical approximation of QED₂. In particular one finds that the contributions of the mass perturbation cancel on the left hand side of (7.144). Thus one can draw Lesson 4.

Lesson 4 :

The masses of the particles that correspond to the Cartan currents obey the Witten-Veneziano formula (7.144).

It has to be remarked that (7.144) is also a verification of the original form of the Witten-Veneziano formula, because the topological susceptibility of the quenched theory reduces to the contact term. It is not true, however, that the topological susceptibility appearing in the formula expresses a property of the long distance fluctuations of the topological density.

Chapter 8

Confinement in the massless model

In this chapter the problem of confinement in the massless model with $g = 0$ will be addressed. I consider a generalization of the confinement criterion proposed by Fredenhagen and Marcu [24]. The original proposal (for lattice-QCD) is to study a sequence of dipole states

$$|\Phi_{\vec{x},\vec{y}}\rangle = \sum_{i,\alpha,\beta} \psi_{i,\alpha}(\vec{x}) U_{\alpha\beta}(\mathcal{C}_{\vec{x},\vec{y}}) \bar{\psi}_{i,\beta}(\vec{y}) |0\rangle . \quad (8.1)$$

i is a flavor index, and α and β are Dirac indices. $U_{\alpha\beta}(\mathcal{C}_{\vec{x},\vec{y}})$ is the path ordered integral of the gauge field along the path $\mathcal{C}_{\vec{x},\vec{y}}$ which connects the points in space \vec{x} and \vec{y} . It is chosen as a rectangle in the upper Euclidean time half plane (compare Figure 8.1 for the 2d case).

If quark fragmentation occurs for $|\vec{x} - \vec{y}| \rightarrow \infty$, the transition probability of $|\Phi_{\vec{x},\vec{y}}\rangle$ into hadronic states (including the vacuum) should go to 1. In particular since all hadronic states are local excitations of the vacuum one expects

$$\lim_{|\vec{x}-\vec{y}|\rightarrow\infty} \frac{|\langle 0|\Phi_{\vec{x},\vec{y}}\rangle|^2}{||\Phi_{\vec{x},\vec{y}}||^2} = \text{const} \neq 0 . \quad (8.2)$$

If the limit (8.2) is zero, this is an indication that the sequence of dipole states becomes orthogonal to all hadron states and therefore approximates an isolated quark. In [24] it is discussed that the denominator in (8.2) should be replaced by a Wilson loop if the order parameter is computed in the continuum in order to avoid singularities from quark fields at coinciding points. It is known from experiment that a quark-antiquark system is confined. Thus one has a guideline how to construct a proper confinement criterium for QCD which is our best theory for strong interactions. In QED₂ there is no such hint. Thus I consider the following, more general order parameter. Define for $0 \leq l \leq N$

$$\rho^{(l)} := \frac{\left| \left\langle N^{(l)}(L) \right\rangle_0^\theta \right|^2}{\left\langle W^{(l)}(L) \right\rangle_0^\theta} , \quad (8.3)$$

where

$$N^{(l)}(L) := \sum_{\{\alpha_a, \beta_a\}} \prod_{b=1}^l \left[\psi_{\alpha_b}^{(b)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_{\beta_b}^{(b)}(+L, 0) \right] , \quad (8.4)$$

with the gauge field transporter $U(\mathcal{C}(L))$ defined as

$$U(\mathcal{C}(L)) := \exp \left(i \int_{\mathcal{C}(L)} e A_\mu(x) dx_\mu \right) . \quad (8.5)$$

Finally $W^{(m)}(L)$ denotes the Wilson loop

$$W^{(l)}(L) := \exp \left(il \int_{\mathcal{C}^W(L)} e A_\mu(x) dx_\mu \right) . \quad (8.6)$$

The contours $\mathcal{C}(L)$ for the gauge field transporter and $\mathcal{C}^W(L)$ for the Wilson loop are given in Figure 8.1.

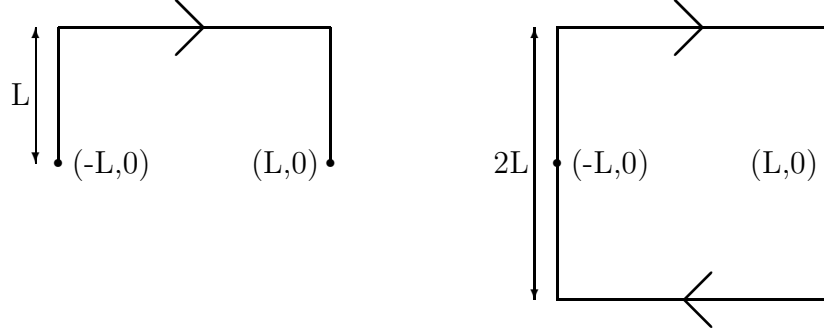


Figure 8.1 : The contours for the gauge field transporter and the Wilson loop.

The contour integrals showing up in (8.5) and (8.6) can be rewritten in terms of scalar products (insert $\mathcal{C}(L)$ and $\mathcal{C}^W(L)$ for \mathcal{K})

$$\int_{\mathcal{K}(L)} A_\mu(x) dx_\mu = (A_\mu, j_{\mathcal{K}\mu}^{(L)}) , \quad (8.7)$$

where the current $j_{\mathcal{K}}$ has its support on the contour \mathcal{K} . In particular the current for the gauge field transporter is given by

$$j^{(L)}(x) = \begin{pmatrix} \theta(x_1 + L)\theta(L - x_1) \delta(x_2 - L) \\ \theta(x_2)\theta(L - x_2) [\delta(x_1 + L) - \delta(x_1 - L)] \end{pmatrix} . \quad (8.8)$$

The current for the Wilson loop reads

$$j_W^{(L)}(x) = \begin{pmatrix} \theta(x_1 + L)\theta(L - x_1) [\delta(x_2 - L) - \delta(x_2 + L)] \\ \theta(x_2 + L)\theta(L - x_2) [\delta(x_1 + L) - \delta(x_1 - L)] \end{pmatrix}. \quad (8.9)$$

Later I will need the Fourier transform of the currents given by (compare (B.3))

$$\hat{j}^{(L)}(p) = \frac{1}{\pi} \sin(p_1 L) \begin{pmatrix} \frac{e^{-ip_2 L}}{p_1} \\ \frac{1 - e^{-ip_1 L}}{p_2} \end{pmatrix}, \quad (8.10)$$

and

$$\hat{j}_W^{(L)}(p) = \frac{2i}{\pi} \sin(p_1 L) \sin(p_2 L) \begin{pmatrix} -\frac{1}{p_1} \\ +\frac{1}{p_2} \end{pmatrix}. \quad (8.11)$$

I start with the evaluation of the Wilson loop.

$$\langle W^{(l)}(L) \rangle_0^\theta = \int d\mu_Q[A] \exp(i l e(A, j_W^{(L)})) = \exp\left(-\frac{1}{2} l^2 e^2 (j_W^{(L)}, Q j_W^{(L)})\right). \quad (8.12)$$

The covariance Q for the gauge field is given by (4.45). Fourier transforming it and inserting (8.11) gives for the Wilson loop

$$\langle W^{(l)}(L) \rangle_0^\theta = \exp\left(-\frac{1}{2} l^2 e^2 \frac{4}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{p^2}{p_1^2 p_2^2} \sin^2(p_1 L) \sin^2(p_2 L)\right), \quad (8.13)$$

where the dynamically generated mass m_d reads ($g = 0$)

$$m_d := e \sqrt{\frac{N}{\pi}}. \quad (8.14)$$

Before one starts to evaluate the numerator of (8.3) one first has to check which case of the θ -prescription has to be applied (compare (5.53)). According to the $U(1)_A$ transformation defined in (5.50) one finds that

$$\prod_{a=1}^l \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(a)}(L, 0) \right], \quad (8.15)$$

remains invariant, and the θ -prescription reduces to the naive expectation value as can be seen from (5.53). Of course this is only one of the terms showing up in the sum for $N^{(l)}(L)$. It will turn out that it is the crucial contribution that determines the confinement behaviour. Thus I start my analysis with this expression, and discuss the other terms later.

$$\left\langle \prod_{a=1}^l \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(a)}(L, 0) \right] \right\rangle_0^\theta$$

$$= \int d\mu_Q[A] \exp \left(ile(A, j^{(L)}) \right) \prod_{a=1}^l G_{12}((-L, 0), (+L, 0); A) . \quad (8.16)$$

Using (4.53) to rewrite A_μ in terms of the scalar field φ one obtains (insert (4.58) for the propagator)

$$\begin{aligned} & \left\langle \prod_{a=1}^l \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(a)}(L, 0) \right] \right\rangle_0^\theta \\ &= \frac{1}{(2\pi)^l} \left(\frac{-1}{2L} \right)^l \int d\mu_{\tilde{Q}}[\varphi] \exp \left(-le \left(\varphi, \delta(-L, 0) - \delta(L, 0) - i\varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)} \right) \right) \\ &= \frac{1}{(2\pi)^l} \left(\frac{-1}{2L} \right)^l \exp \left(\frac{1}{2} l^2 e^2 \left([\delta(-L, 0) - \delta(L, 0)], \tilde{Q}[\delta(-L, 0) - \delta(L, 0)] \right) \right) \\ & \quad \times \exp \left(-\frac{1}{2} l^2 e^2 \left(\varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)}, \tilde{Q} \varepsilon_{\rho\sigma} \partial_\sigma j_\rho^{(L)} \right) \right) \\ & \quad \times \exp \left(-\frac{i}{2} l^2 e^2 \text{2Re} \left(\varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)}, \tilde{Q} [\delta(-L, 0) - \delta(L, 0)] \right) \right) . \quad (8.17) \end{aligned}$$

The convolution with the δ -functional $\delta(x)$ with support at the space-time point x is understood as $(t, \delta(x)) = t(x)$. Again the scalar products appearing in (8.17) will be rewritten as momentum space integrals. The Fourier transform of the δ -functional is given by (compare (B.3))

$$[\hat{\delta}(-L, 0) - \hat{\delta}(L, 0)](p) = \frac{i}{\pi} \sin(p_1 L) . \quad (8.18)$$

Using this and (8.8) and Equation (4.54) for the covariance \tilde{Q} , the term that mixes the current $j^{(L)}$ and the δ -functionals can be written as

$$\begin{aligned} & \left(\varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)}, \tilde{Q} [\delta(-L, 0) - \delta(L, 0)] \right) \\ &= \frac{1}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{1}{p^2} \sin^2(p_1 L) \left[\frac{p_2}{p_1} e^{ip_2 L} - \frac{p_1}{p_2} (1 - e^{ip_2 L}) \right] = 0 . \quad (8.19) \end{aligned}$$

In the last step I used that the integrand is odd in p_1 to show that this term vanishes. Rewriting the other two exponents in (8.17) in terms of momentum space integrals, one ends up with

$$\begin{aligned} & \left\langle \prod_{a=1}^l \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(a)}(L, 0) \right] \right\rangle_0^\theta \\ &= \frac{1}{(2\pi)^l} \left(\frac{-1}{2L} \right)^l \exp \left(\frac{1}{2} l^2 e^2 \frac{1}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{1}{p^2} \sin^2(p_1 L) \right) \end{aligned}$$

$$\times \exp \left(-\frac{1}{2} l^2 e^2 \frac{1}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{1}{p^2} \sin^2(p_1 L) \left[\frac{p_2^2}{p_1^2} + 2 \frac{p^2}{p_2^2} (1 - \cos(p_2 L)) \right] \right) . \quad (8.20)$$

Considering the contribution coming from (8.16) alone, one obtains

$$\begin{aligned} \rho_{12}^{(l)}(L) &:= \frac{\left| \left\langle \prod_{a=1}^l \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(a)}(L, 0) \right] \right\rangle_0^\theta \right|^2}{\langle W^{(l)}(L) \rangle_0^\theta} \\ &= \frac{1}{(2\pi)^{2l}} \left(\frac{1}{2L} \right)^{2l} \exp \left(l^2 I_1 + l^2 I_2 + l^2 I_3 \right) , \end{aligned} \quad (8.21)$$

with I_1 from the Wilson loop result (8.13)

$$I_1 := e^2 \frac{2}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{p^2}{p_1^2 p_2^2} \sin^2(p_1 L) \sin^2(p_2 L) \quad (8.22)$$

and I_2 from (8.20)

$$I_2 := e^2 \frac{1}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{1}{p^2} \sin^2(p_1 L) , \quad (8.23)$$

and finally I_3 which stems from the second exponent in (8.20)

$$I_3 := -e^2 \frac{1}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{1}{p^2} \sin^2(p_1 L) \left[\frac{p_2^2}{p_1^2} + 2 \frac{p^2}{p_2^2} (1 - \cos(p_2 L)) \right] . \quad (8.24)$$

Using trigonometric identities and the symmetry of some terms in the integrands under the interchange $p_1 \leftrightarrow p_2$ one can show

$$I_1 + I_3 = I_2 + I_R , \quad (8.25)$$

where I_R is given by

$$\begin{aligned} I_R &= \frac{e^2}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{1}{p_2^2} \left(\sin^2(p_2 L) - 2 \sin^2(p_2 L/2) \right) \\ &- 2 \frac{e^2}{\pi^2} \int d^2 p \frac{1}{p^2 + m_d^2} \frac{1}{p_2^2} \cos(p_1 2L) \left(\sin^2(p_2 L) - \sin^2(p_2 L/2) \right) . \end{aligned} \quad (8.26)$$

Using Formula (B.9) from Appendix B.1, one can solve I_2 (insert m_d)

$$I_2 := \frac{1}{N} \left[\ln(2L) + \ln \left(\frac{e}{2} \sqrt{\frac{N}{\pi}} \right) + K_0 \left(2Le \sqrt{\frac{N}{\pi}} \right) + \gamma \right] . \quad (8.27)$$

I_R was not solved explicitly, but in the Appendix B.1 the behaviour

$$I_R = \frac{1}{N} + O(e^{-L}) \quad \text{for } L \rightarrow \infty , \quad (8.28)$$

is established (see (B.12)). Thus when putting things together, one concludes

$$\begin{aligned}\rho_{12}^{(l)}(L) &= \frac{1}{(2\pi)^{2l}} \left(\frac{1}{4L^2}\right)^l (4L^2)^{\frac{l^2}{N}} \left(\frac{e^2 N}{4\pi} e^{2\gamma+1}\right)^{\frac{l^2}{N}} \exp(O(e^{-L})) \\ &= \left(\frac{1}{4L^2}\right)^{l(1-\frac{1}{N})} \frac{1}{(2\pi)^{2l}} \left(\frac{e^2 N}{4\pi} e^{2\gamma+1}\right)^{\frac{l^2}{N}} \exp(O(e^{-L})) .\end{aligned}\quad (8.29)$$

Inspecting (8.17)-(8.20) one finds that

$$\prod_{a=1}^l \left[\psi_2^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_1^{(a)}(L, 0) \right] , \quad (8.30)$$

gives the same result as was obtained for (8.17). Thus taking into account all possible combinations of the terms (8.17) and (8.30) in $\rho^{(l)}(L)$ one obtains an extra factor 4 compared to (8.29). Multiplying $\rho_{12}^{(l)}(L)$ with this extra factor 4, gives already the final result, since all other possible contributions vanish with an extra power of $1/L$ for $L \rightarrow \infty$. This can be seen as follows. Consider e.g. ($l \geq 2$)

$$\begin{aligned}& \left\langle \psi_2^{(1)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_1^{(1)}(L, 0) \prod_{a=2}^l \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(a)}(L, 0) \right] \right\rangle_0^\theta \\ &= \int d\mu_Q[\varphi] \exp\left(ile\left(\varphi, \varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)}\right)\right) \\ &\times G_{21}\left((-L, 0), (+L, 0); \varphi\right) \prod_{a=2}^l G_{12}\left((-L, 0), (+L, 0); \varphi\right) \\ &= \frac{1}{(2\pi)^{2l}} \left(\frac{-1}{2L}\right)^l d\mu_Q[\varphi] \exp\left(ile\left(\varphi, \varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)}\right)\right) \\ &\times \exp\left(-[l-2]e\left(\varphi, \delta(-L, 0) - \delta(L, 0)\right)\right) ,\end{aligned}\quad (8.31)$$

where I used the exponential dependence (4.58) on the external field φ of the propagator in the last step. Comparing (8.17) and (8.31) one finds that the latter expression has only the factor $l-2$ in front of the exponent containing the δ -functionals. After evaluation of the functional integrals this amounts to an extra factor ($l \geq 2$)

$$\exp\left([-l^2 + (l-2)^2] \frac{I_2}{2}\right) \sim \left(\frac{1}{2L}\right)^{\frac{2}{N}(l-1)} . \quad (8.32)$$

Thus all terms of the type (8.31) where the spinor indices do not all assume the same value are suppressed by the extra factor (8.32). Finally there are

some more possible contributions that were not discussed so far. They show up only for $l = N$ when clustering can be violated in principle and a different case of the θ -prescription (5.53) has to be used. A typical contribution of that type is

$$\begin{aligned} & \left\langle \prod_{a=1}^N \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_1^{(a)}(L, 0) \right] \right\rangle_0^\theta \\ &= e^{i\theta} \lim_{\tau \rightarrow \infty} \left\langle \mathcal{O}_-^a(y + \hat{\tau}) \prod_{a=N}^l \left[\psi_1^{(a)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_1^{(a)}(L, 0) \right] \right\rangle_0, \end{aligned} \quad (8.33)$$

where I already inserted the θ -prescription (5.52). Due to the vanishing diagonal entries of the propagator (see (4.58)) the $\psi_1^{(a)}(-L, 0)$ and $\bar{\psi}_1^{(a)}(L, 0)$ can only contract to \mathcal{O}_-^a , and the terms

$$\exp \left(\pm e \left(\varphi, \delta(-L, 0) - \delta(L, 0) \right) \right), \quad (8.34)$$

cannot emerge. As for the term (8.31) one concludes that (8.33) acquires an extra factor

$$\left(\frac{1}{2L} \right)^{\frac{l^2}{2}}. \quad (8.35)$$

The final result thus reads

$$\begin{aligned} & \rho^{(l)}(L) \\ &= \left(\frac{1}{4L^2} \right)^{l(1-\frac{1}{N})} \frac{4}{(2\pi)^{2l}} \left(\frac{e^2}{4} \frac{N}{\pi} e^{2\gamma+1} \right)^{\frac{l^2}{N}} \exp(O(e^{-L})) \left[1 + O \left(\left(\frac{1}{2L} \right)^{\frac{2}{N}(l-1)} \right) \right]. \end{aligned} \quad (8.36)$$

Obviously the damping factor can only be switched off by setting $l = N$, and one ends up with¹

$$\rho^{(l)} = \lim_{L \rightarrow \infty} \rho^{(l)}(L) = \begin{cases} \frac{4}{(2\pi)^{2N}} \left(\frac{e^2 N}{4\pi} \right)^N e^{N(2\gamma+1)} & \text{for } l = N \\ 0 & \text{for } l < N \end{cases}. \quad (8.37)$$

Thus in the N -flavor model an arrangement of N ‘quarks’ is bound by a confining force to an arrangement of N ‘antiquarks’.

A generalization of the order parameter (8.3) shows that an operator of $N + l$ with $0 < l < N$ behaves like the product of only l quarks². Define

$$\rho^{(N+l)} := \frac{\left| \left\langle N^{(N+l)}(L) \right\rangle_0^\theta \right|^2}{\left\langle W^{(N+l)}(L) \right\rangle_0^\theta}, \quad (8.38)$$

¹Up to an overall factor the result (8.37) for $N = 1$ flavor can be found in [3].

² In general products of the type (8.39) with n factors always behaves as the $l = N \bmod(N)$ expression (8.3).

where

$$\begin{aligned}
N^{(N+l)}(L) &:= \sum_{\{\alpha_a, \beta_a, \alpha'_a, \beta'_a\}} \left\{ \prod_{b=1}^N \left[\psi_{\alpha_b}^{(b)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_{\beta_b}^{(b)}(+L, 0) \right] \right. \\
&\quad \times \left. \prod_{c=1}^l \left[\psi_{\alpha'_c}^{(c)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_{\beta'_c}^{(c)}(+L, 0) \right] \right\}, \quad (8.39)
\end{aligned}$$

$W^{(N+l)}(L)$ is obtained by replacing $l \rightarrow N + l$ in (8.6). Again I start with considering a special arrangement of the spinor indices

$$\begin{aligned}
&\left\langle \prod_{b=1}^N \left[\psi_1^{(b)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(b)}(+L, 0) \right] \prod_{c=1}^l \left[\psi_1^{(c)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(c)}(+L, 0) \right] \right\rangle_0^\theta \\
&= \int d\mu_{\tilde{Q}}[\varphi] \exp \left(i[N + l] e \left(\varphi, \varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)} \right) \right) \\
&\quad \times \prod_{b=1}^l \left[G_{12}((-L, 0), (L, 0); \varphi) - G_{12}((-L, 0), (L, 0); \varphi) \right] \\
&\quad \times \prod_{c=l+1}^N G_{12}((-L, 0), (L, 0); \varphi) = 0, \quad (8.40)
\end{aligned}$$

where I used the factorization with respect to the flavors, and rewrote the expectation values $\langle \psi_1^{(a)}(-L, 0) \bar{\psi}_2^{(a)}(L, 0) \psi_1^{(a)}(-L, 0) \bar{\psi}_2^{(a)}(L, 0) \rangle_0$ for fixed a in terms of propagators. Analyzing other possible arrangements of the spinor indices one finds that the crucial terms take the form

$$\begin{aligned}
&\left\langle \prod_{b=1}^N \left[\psi_1^{(b)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_1^{(b)}(+L, 0) \right] \prod_{c=1}^l \left[\psi_2^{(c)}(-L, 0) U(\mathcal{C}(L)) \bar{\psi}_2^{(c)}(+L, 0) \right] \right\rangle_0^\theta \\
&= \int d\mu_{\tilde{Q}}[\varphi] \exp \left(i[N + l] e \left(\varphi, \varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)} \right) \right) \\
&\quad \prod_{b=1}^l \left[-G_{12}((-L, 0), (L, 0); \varphi) G_{21}((-L, 0), (L, 0); \varphi) \right] \prod_{c=l+1}^N G_{12}((-L, 0), (L, 0); \varphi) \\
&= (-1)^N \frac{1}{(2\pi)^{N+l}} \left(\frac{1}{2L} \right)^{N+l} \int d\mu_{\tilde{Q}}[\varphi] \exp \left(i[N + l] e \left(\varphi, \varepsilon_{\mu\nu} \partial_\nu j_\mu^{(L)} \right) \right) \\
&\quad \times \exp \left(-[N - l] e \left(\varphi, \delta(-L, 0) - \delta(L, 0) \right) \right) \\
&= (-1)^N \frac{1}{(2\pi)^{N+l}} \left(\frac{1}{2L} \right)^{N+l} \exp \left(\frac{1}{2} [N - l]^2 I_2 + \frac{1}{2} [N + l]^2 I_3 \right). \quad (8.41)
\end{aligned}$$

Considering again the contributions that stem from terms (8.41) alone, one obtains the behaviour

$$\begin{aligned}
&\sim \left(\frac{1}{2L}\right)^{2[N+l]} \exp\left([N-l]^2 I_2 + [N+l]^2 (I_1 + I_3)\right) \\
&\sim \left(\frac{1}{2L}\right)^{2[N+l]} \exp\left(\left([N-l]^2 + [N+l]^2\right) I_2\right) \\
&\sim \left(\frac{1}{2L}\right)^{2[N+l] - \frac{1}{N}([N-l]^2 + [N+l]^2)} = \left(\frac{1}{4L^2}\right)^{l(1-\frac{1}{N})} .
\end{aligned} \tag{8.42}$$

Arguments similar to the discussion for the simpler case $\rho^{(l)}$ show that terms that are not of the type (8.41) acquire extra powers of $1/L$, and thus do not influence the general behaviour. One ends up with

$$\rho^{(N+l)} = \lim_{L \rightarrow \infty} \rho^{(N+l)}(L) = \begin{cases} \text{const} \neq 0 & \text{for } l = 0, N \\ 0 & \text{for } 0 < l < N \end{cases} . \tag{8.43}$$

The physical interpretation suggested by this behaviour of the Fredenhagen-Marcu order parameters (8.37) and (8.43) is the following: The model has N distinct superselection sectors labeled by a charge Q that is defined only modulo N . To obtain a state in the sector of charge $Q = n$, $n < N$, one applies an operator consisting of n ‘quarks’ and n antiquarks, separated by distance $2L$ and takes the limit $L \rightarrow \infty$.

A rather curious result is obtained, when one computes $\rho^{(l)}(L)$ defined in (8.3) in the model with finite g , i.e. with the Thirring term coupled. One finds

$$\lim_{L \rightarrow \infty} \rho^{(l)}(L) = \begin{cases} \infty & \text{for } l = N \\ 0 & \text{for } 0 < l < N \end{cases} . \tag{8.44}$$

This result might be related to problems with OS-positivity. It turns out that one recovers a finite constant (and thus the problems with OS-positivity vanish), when the auxiliary field is transported along the contours as well. In particular $U(\mathcal{C}(L))$ defined in (8.5) is replaced by

$$\exp\left(i \int_{\mathcal{C}(L)} \left[e A_\mu(x) + \sqrt{g} h_\mu(x)\right] dx_\mu\right) , \tag{8.45}$$

and similar for the Wilson loop (8.6).

Summary

Using the method of Euclidean path integrals, QED₂ with mass and N flavors of fermions has been investigated. In order to use the explicitly known determinant for massless fermions, the expectation functional was expanded with respect to the fermion masses. It has been argued that it does not make sense to expand the determinant directly since all involved terms behave $\propto m^2 \ln m$ in infinite volume. A Thirring term has been included in order to make the short distance singularities which show up in the mass perturbation integrable. It can be produced by an auxiliary field which couples in the same way as the gauge field does. Using the Gaussian behaviour of fermion determinant and action, the formally defined path integral was given a mathematically precise meaning in terms of Gaussian functional integrals.

Evaluation of a general ansatz allows the identification of operators that violate clustering in the massless model. It turned out that the cluster decomposition property is violated by operators that are singlets under $U(1)_V \times SU(N)_L \times SU(N)_R$ but transform nontrivially under $U(1)_A$. The nontrivial transformation properties under $U(1)_A$ were used to decompose the expectation functional into clustering θ -vacua. The original vacuum state was shown to be a mixture of the θ -vacua.

A generalized generating functional was used to bosonize the currents corresponding to a Cartan subalgebra of $U(N)$ together with the chiral densities. It was shown that for vanishing fermion masses the Cartan currents can be bosonized in terms of one massive and $N-1$ massless scalar fields. It was demonstrated that no bosonization in terms of local scalar fields exists for the whole set of N^2 currents corresponding to all generators of $U(N)$. Nevertheless it was possible to show that the Hilbert space for all N^2 classically conserved currents is a tensor product of the Hilbert space for the $U(1)$ -current (which is the Fock space of a massive free field) with the Hilbert space of $N^2 - 1$ currents constructed out of free massless fermions.

Summing up the mass perturbation series, the Cartan currents were bosonized also for nonvanishing fermion masses. The corresponding scalar theory turned out to be a generalization of the Sine-Gordon model. The mass perturbation series was shown to converge when imposing a space-time cutoff. By evaluating explicitly the first few terms of the series it was demonstrated that removing the cutoff termwise is only possible for the one flavor model. It was

argued that the correct treatment (sum up the series and remove the cutoff nonperturbatively) requires some new mathematical methods. Since the space time cutoff spoils translation invariance which is necessary for the computation of self energies, one is reduced to a semiclassical approximation in order to compute the mass spectrum of the bosonized currents. Nevertheless for vanishing fermion masses the semiclassical approximation is exact, and thus is expected to give good results for small masses.

A generalization of the Fredenhagen Marcu order parameter was evaluated in order to investigate the confinement properties of the massless model. It turned out that in the N flavor model an arrangement of N quarks is bound by a confining force to an arrangement of N antiquarks.

So far for the construction of QED_2 . The whole investigation was motivated by a critical survey of three topics from QCD which are closely related to each other. Namely the construction of θ -vacua from topologically nontrivial sectors, the $U(1)$ -problem and Witten-Veneziano formulas. Those three subjects can be modelled rather well in QED_2 . The idea is to circumvent poorly defined concepts like the superposition of topologically nontrivial sectors to a θ -vacuum, and to learn from the construction summarized above. This enterprise lead to the following four lessons for QED_2 .

Lesson 1 : (page 51)

The structure of the vacuum functional that has been suggested within the instanton picture is recovered.

In particular only operators with chirality $2N\nu$, $\nu \in \mathbb{Z}$ have nonvanishing vacuum expectation values, as has been claimed by 't Hooft for QCD.

Lesson 2 : (page 66)

The axial $U(1)$ -symmetry is not a symmetry on the physical Hilbert space, and there is no $U(1)$ -problem for QED_2 .

The same should be true for QCD since it is doubtful if the generator for the $U(1)$ -axial symmetry really exists on the physical Hilbert space.

Lesson 3 : (page 74)

Physics does not depend on θ if at least one of the fermion masses vanishes.

This property is commonly believed to hold for QCD as well.

Lesson 4 : (page 102)

The masses of the particles that correspond to the Cartan currents obey a Witten-Veneziano type formula.

Witten-Veneziano formulas were also derived for QCD (see the discussion in

Section 2.3 for their status).

To sum up, several interesting insights into the three ‘mysteries’ as they show up in QED_2 have been obtained. It is hoped that this helps to come to a better understanding of their QCD counterparts.

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Appendix A

The field theory appendix

This appendix contains the ingredients from (two dimensional) Euclidean quantum field theory I am going to use. All the material is well known, but distributed over various textbooks. The appendix summarizes the formulas and fixes the notation.

A.1 Propagators in two dimensions

In this section the expressions for various two dimensional Euclidean propagators that will be used in the main part are summarized.

Free massless bosons:

The defining equation for the Green's functions reads

$$-\Delta C_0(x-y) = \delta(x-y) , \quad (\text{A.1})$$

and is solved by

$$C_0(x) = -\frac{1}{4\pi} \ln(\mu^2 x^2) . \quad (\text{A.2})$$

μ^2 is an arbitrary constant (compare the appendix on Wick ordering). The above solution is understood in the sense of distributions. After smearing with a test function t

$$(-\Delta C_0, t) \equiv (C_0, -\Delta t) = t(0) = (\delta, t) . \quad (\text{A.3})$$

It can be found in e.g. [29]. It has to be remarked that a massless scalar field φ in two dimensions does not define a proper Wightman field theory [19], but $\partial_\mu \varphi$ does. Only the latter will be used here, and one has to take derivatives of the formal propagator (A.2) which remove the dependence of the results on μ .

Free massive bosons:

The defining equation reads

$$(-\Delta + m^2)C_m(x-y) = \delta(x-y) . \quad (\text{A.4})$$

It can be solved by Fourier transformation

$$C_m(x) = \frac{1}{(2\pi)^2} \int d^2k \frac{e^{ikx}}{k^2 + m^2} = \frac{1}{2\pi} K_0(m|x|) . \quad (\text{A.5})$$

The momentum space integral is to be interpreted in distributional sense again (see [29]) and can be found in Appendix B.2. It has the following short and long distance behaviour.

$$C_m(x) = \begin{cases} -\frac{1}{2\pi} \left(\ln \left(\frac{m|x|}{2} \right) + \gamma + O(x^2) \right) & \text{for } x \rightarrow 0 \\ \frac{1}{2\pi} \left(\frac{\pi}{2m|x|} \right)^{\frac{1}{2}} e^{-m|x|} \left(1 + O\left(\frac{1}{x}\right) \right) & \text{for } x \rightarrow \infty \end{cases} \quad (\text{A.6})$$

Free massless fermions:

The fermion propagator can be constructed from the boson propagator. It has to obey

$$\gamma_\mu \partial_\mu G^o(x - y) = \delta(x - y) . \quad (\text{A.7})$$

Using $\not{\partial} \not{\partial} = \Delta$ and the boson propagator one finds the solution

$$G^o(x) = -\gamma_\mu \partial_\mu C_0(x) = \frac{1}{2\pi} \frac{\gamma_\mu x_\mu}{x^2} . \quad (\text{A.8})$$

Massless fermions in an external field:

The Green's function equation reads

$$\gamma_\mu (\partial_\mu - iB_\mu(x)) G(x, y; B) = \delta(x - y) . \quad (\text{A.9})$$

The solution is related to the free propagator $G^o(x)$

$$G(x, y; B) = G^o(x - y) e^{i[\Phi(x) - \Phi(y)]} , \quad (\text{A.10})$$

where

$$\Phi(x) = - \int d^2z D(x - z) (\partial_\mu B_\mu(z) + i\gamma_5 \varepsilon_{\mu\nu} \partial_\mu B_\nu(z)) . \quad (\text{A.11})$$

By direct evaluation it is easy to show that $\partial_\mu \Phi(x) = B_\mu(x)$. It has to be remarked that the inversion of the Laplace operator requires some mild regularity and falloff properties of the external field B_μ . In particular its Fourier transform at zero momentum has to vanish. This corresponds to zero winding (compare Section 3.3).

In the main part I will only work with transverse fields obeying $\partial_\mu B_\mu(x) = 0$. In this case Φ reduces to

$$\Phi(x) = -i\gamma_5 \int d^2z D(x - z) \left(+ i\gamma_5 \varepsilon_{\mu\nu} \partial_\mu B_\nu(z) \right) = i\gamma_5 \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} B_\nu(x) . \quad (\text{A.12})$$

$e^{i[\Phi(x)-\Phi(y)]}$ can be evaluated easily and gives the diagonal matrix in spin space

$$e^{i[\Phi(x)-\Phi(y)]} = \text{diag}\left(e^{-[\chi(x)-\chi(y)]}, e^{+[\chi(x)-\chi(y)]}\right), \quad (\text{A.13})$$

where I defined

$$\chi(x) := \frac{\varepsilon_{\mu\nu}\partial_\mu}{\Delta} B_\nu, \quad \tilde{x} := x_1 + ix_2. \quad (\text{A.14})$$

Performing the matrix multiplication one ends up with

$$G(x, y; B) = \frac{1}{2\pi} \frac{1}{(x-y)^2} \begin{pmatrix} 0 & e^{-[\chi(x)-\chi(y)]} \overline{(\tilde{x}-\tilde{y})} \\ e^{+[\chi(x)-\chi(y)]} (\tilde{x}-\tilde{y}) & 0 \end{pmatrix}. \quad (\text{A.15})$$

A.2 Gaussian measures

The aim of this appendix is to give a taste of the mathematics of Gaussian functional integrals, and to introduce the notations used in the main part. A nice introduction to the topic can be found in [50], the mathematical details are discussed in [31].

Gaussian measures are measures on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Measureable sets can be constructed by considering *cylinder sets* Z defined in the following way. Let t_1, \dots, t_n be a fixed set of test functions in $\mathcal{S}(\mathbb{R}^d)$ and B a Borel set in \mathbb{R}^n . The set

$$Z := \left\{ \varphi \in \mathcal{S}'(\mathbb{R}^d) \mid (\varphi(t_1), \dots, \varphi(t_n)) \in B \right\}, \quad (\text{A.16})$$

is called a cylinder set generated by t_1, \dots, t_n , with basis B . Equation (A.16) already shows that measures on the cylinder sets can be defined by making use of the measurability of B .

The second ingredient for the construction of Gaussian measures are *covariance operators*. A continuous, positive, linear map C from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ which is nondegenerate

$$(t, Ct)_{L^2} = 0 \quad \text{only for } t = 0, \quad (\text{A.17})$$

is called a covariance operator.

Now one can define the *Gaussian measure with covariance C* of a cylinder set Z by

$$\mu_C[Z] := \int_{\varphi \in Z} d\mu_C[\varphi], \quad (\text{A.18})$$

$$d\mu_C[\varphi] := \frac{1}{\sqrt{\det(2\pi\tilde{C})}} \exp\left(-\frac{1}{2} \sum_{i,j=1}^n \alpha_i (\tilde{C}^{-1})_{i,j} \alpha_j\right) \prod_{l=1}^n d\alpha_l, \quad (\text{A.19})$$

where

$$\tilde{C}_{i,j} := (t_i, Ct_j)_{L^2} , \quad (\text{A.20})$$

and $\prod_{l=1}^n d\alpha_l$ denotes Lebesgue measure which has to be integrated over B the basis of Z .

So far for the constructive aspects of Gaussian measures. To show that (A.18) defines a proper measure on all of $\mathcal{S}'(\mathbb{R}^d)$, some more work has to be done. It has to be established that the cylinder sets can be extended to a Boolean σ -algebra. Furthermore it has to be shown that $\mu_C[Z]$ does not depend on the choice of the generating elements t_1, \dots, t_n or the basis B . Finally it has to be checked that $\mu_C[Z]$ obeys the properties (σ -additivity, regularity,), that allow to extend it to a measure on all of $\mathcal{S}'(\mathbb{R}^d)$. Most of this material can be found in the very explicit books of Gelfand and Shilow (Vilenkin) [29].

I finish this section by quoting the two ‘holy formulas’ of Gaussian integration

$$\int d\mu_C[\varphi] e^{\pm i\varphi(t)} = e^{-\frac{1}{2}(t,Ct)} , \quad (\text{A.21})$$

$$\int d\mu_C[\varphi] e^{\pm \varphi(t)} = e^{+\frac{1}{2}(t,Ct)} . \quad (\text{A.22})$$

Those two results can be obtained easily from (A.18)-(A.20) since it is sufficient to consider the cylinder set with generating element t and basis $B = \mathbb{R}$. The Equations (A.21), (A.22) reduce to ordinary Gaussian integrals then.

A.3 Finite action is zero measure

In this appendix I discuss a toy example which illustrates that field configurations with finite action have measure zero. In fact this is a well known feature which e.g. follows from the properties of Gaussian measures discussed in [18]. The formulation here I borrow from [60].

The system describes infinitely many uncoupled harmonic oscillators. The action is given by

$$S[\{b_j\}] := \frac{1}{2} \sum_{n=1}^{\infty} b_n^2 , \quad b_n \in \mathbb{R} . \quad (\text{A.23})$$

The measure on the space of series $\{b_j\}_{j=1}^{\infty}$ is defined as the product of normalized Gaussian measures, symbolically

$$d\mu[\{b_j\}] := \exp \left(- S[\{b_j\}] \right) \prod_{n=1}^{\infty} \frac{db_n}{\sqrt{2\pi}} , \quad (\text{A.24})$$

and expectation values of operators O acting on $\{b_j\}_{j=1}^{\infty}$ are defined as

$$\langle O \rangle := \int d\mu[\{b_j\}] O[\{b_j\}] . \quad (\text{A.25})$$

From the normalization in (A.24) it follows that

$$\langle 1 \rangle = 1 , \quad (\text{A.26})$$

and hence the measure is a proper probability measure. Now one uses a special observable given by

$$O_N[\{b_j\}] := \exp \left(-\frac{\lambda}{2} \sum_{n=1}^N b_n^2 \right) , \quad \lambda > 0 , \quad (\text{A.27})$$

and considers the limit of $N \in \mathbb{N}$ going to infinity. The evaluation of the expectation value only makes use of Gaussian integrals

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle O_N \rangle &= \lim_{N \rightarrow \infty} \int \prod_{n=1}^{\infty} \frac{db_n}{\sqrt{2\pi}} \exp \left(-\frac{1+\lambda}{2} \sum_{n=1}^N b_n^2 - \frac{1}{2} \sum_{n=N+1}^{\infty} b_n^2 \right) \\ &= \lim_{N \rightarrow \infty} \prod_{n=1}^N \frac{1}{\sqrt{1+\lambda}} = 0 . \end{aligned} \quad (\text{A.28})$$

Using Fatou's lemma (see Vol.1 of [49]) one obtains

$$\langle O_{\infty} \rangle := \langle \lim_{N \rightarrow \infty} O_N \rangle \leq \lim_{N \rightarrow \infty} \langle O_N \rangle = 0 , \quad (\text{A.29})$$

one has to conclude from (A.25) that $O_{\infty}[\{b_j\}] = 0$ almost everywhere, and from the special choice (A.27) for the observable then follows

$$S[\{b_j\}] = \infty \quad \text{almost everywhere} . \quad (\text{A.30})$$

This concludes the toy model discussion. When investing a little bit of time, the same can be shown for the Gaussian integrals of the last section as well. One could define the observable O_N to be

$$O_N[\varphi] := \exp \left(-\frac{\lambda}{2} (\varphi, P_N C^{-1} P_N \varphi) \right) , \quad (\text{A.31})$$

where P_N is the projector on the first N eigenvectors of the covariance operator C . Then one can essentially repeat the arguments given above.

A.4 Wick ordering and massless particles

In this appendix Wick ordering of massive as well as massless bosons is discussed, and the neutrality condition for the massless case is proven. An introduction to the topic can be found in [31].

First I consider two dimensional, Euclidean, massive fields with covariance

$$C_m := (-\Delta + m^2)^{-1} . \quad (\text{A.32})$$

The corresponding Green's function which I denote by the same symbol but an extra space-time argument is given by (compare the propagator appendix)

$$C_m(x - y) = \frac{1}{2\pi} K_0(m|x - y|) . \quad (\text{A.33})$$

The Wick ordering of an exponential with respect to mass M is defined as

$$: e^{i\varphi(f)} :_M \equiv \frac{e^{i\varphi(f)}}{\langle e^{i\varphi(f)} \rangle_{C_M}} = e^{i\varphi(f) + \frac{1}{2}(f, C_M f)} . \quad (\text{A.34})$$

Here φ is a real scalar field with covariance C_m and f denotes a test function in Schwartz space $\mathcal{S}(\mathbb{R})$. With this definition one has for example

$$\begin{aligned} \langle : e^{i\varphi(f)} :_M e^{i\varphi(g)} :_M \rangle_{C_m} &= \exp \left(-\frac{1}{2}(f, C_m f) - (f, C_m g) - \frac{1}{2}(g, C_m g) + \right. \\ &\quad \left. \frac{1}{2}(f, C_M f) + \frac{1}{2}(g, C_M g) \right) . \end{aligned} \quad (\text{A.35})$$

Usually massive fields are normal ordered with respect to their own mass which simplifies the last equation to

$$\langle : e^{i\varphi(f)} :_m e^{i\varphi(g)} :_m \rangle_{C_m} = \exp \left(- (f, C_m g) \right) . \quad (\text{A.36})$$

The test functions f and g can be replaced by δ -sequences leading to e.g.

$$\langle : e^{i\varphi(x)} :_m e^{i\varphi(y)} :_m \rangle_{C_m} = \exp \left(-C_m(x - y) \right) . \quad (\text{A.37})$$

For massless particles the strategy is to Wick order with respect to a given fixed mass M , to take the expectation value with respect to C_m and to perform the limit $m \rightarrow 0$ in the end

$$\langle : e^{i\varphi(f)} :_M e^{i\varphi(g)} :_M \rangle_{C_{m=0}} := \lim_{m \rightarrow 0} \langle : e^{i\varphi(f)} :_M e^{i\varphi(g)} :_M \rangle_{C_m} . \quad (\text{A.38})$$

In the massless limit the *neutrality condition* [20] has to be obeyed in order to obtain nonvanishing expectation values.

Lemma A.1 : Neutrality condition

For test functions t_j , $j = 1, \dots, n$

$$\begin{aligned} &\lim_{m \rightarrow 0} \langle \prod_{j=1}^n : e^{i\varphi(t_j)} :_M \rangle_{C_m} \\ &= \begin{cases} e^{+\frac{1}{2} \sum_{i=1}^n (t_i, C_M t_i)} e^{-\frac{1}{2} \sum_{i \neq j} (t_i, C_0 t_j)} & \text{for } \sum_{j=1}^n q_j = 0 \\ 0 & \text{for } \sum_{j=1}^n q_j \neq 0 \end{cases} , \end{aligned} \quad (\text{A.39})$$

where

$$C_0(x) = -\frac{1}{4\pi} \left(\ln(x^2) + 2\gamma - \ln 4 \right) , \quad (\text{A.40})$$

and

$$q_j := \int d^2x \, t_j(x) . \quad (\text{A.41})$$

Proof :

$$\left\langle \prod_{j=1}^n : e^{i\varphi(t_j)} :_M \right\rangle_{C_m} = e^{\frac{1}{2} \sum_{i=1}^n (t_i, C_M t_i) - \frac{1}{2} \sum_{i,j} (t_i, C_m t_j)} . \quad (\text{A.42})$$

From the propagator appendix I use the short distance (\Leftrightarrow small mass) behaviour (A.6) of the massive boson propagator

$$C_m(z) = -\frac{1}{4\pi} \ln(z^2) - \frac{1}{4\pi} \left(2\gamma + \ln\left(\frac{m^2}{4}\right) + O(m^2 z^2) \right) . \quad (\text{A.43})$$

This implies

$$\begin{aligned} (t_i, C_m t_j) &= \int d^2x \, d^2y \, C_m(x-y) \, t_i(x) \, t_j(y) \\ &= (t_i, C_0 t_j) - \frac{1}{4\pi} \ln(m^2) \, q_i \, q_j + O(m^2) . \end{aligned} \quad (\text{A.44})$$

Inserting this into (A.42) gives

$$\begin{aligned} \left\langle \prod_{j=1}^n : e^{i\varepsilon_j \varphi(x_j)} :_M \right\rangle_{C_m} &= e^{\frac{1}{2} \sum_{i=1}^n (t_i, C_M t_i) - \frac{1}{2} \sum_{i,j} (t_i, C_0 t_j)} \\ &\quad \times m^{\frac{1}{4\pi} (\sum_{i=1}^n q_i)^2} e^{O(m^2)} , \end{aligned} \quad (\text{A.45})$$

which immediately leads to the desired result. \square

Inserting δ -sequences one can find another formulation [25].

Lemma A.2 :

For pairwise disjoint space time arguments x_j , $j = 1, \dots, n$ and real constants ε_j , $j = 1, \dots, n$

$$\begin{aligned} &\lim_{m \rightarrow 0} \left\langle \prod_{j=1}^n : e^{i\varepsilon_j \varphi(x_j)} :_M \right\rangle_{C_m} \\ &= \begin{cases} \left(\frac{1}{M} \right)^{\frac{1}{4\pi} \sum_{j=1}^n \varepsilon_j^2} e^{-\frac{1}{2} \sum_{i \neq j} \varepsilon_i \varepsilon_j C_0(x_i - x_j)} & \text{for } \sum_{j=1}^n \varepsilon_j = 0 \\ 0 & \text{for } \sum_{j=1}^n \varepsilon_j \neq 0 . \end{cases} \end{aligned} \quad (\text{A.46})$$

Proof :

For the proof one only has to insert some δ -sequence and to use

$$\lim_{z \rightarrow 0} (C_M(z) - C_0(z)) = -\frac{1}{4\pi} \ln(M^2), \quad (\text{A.47})$$

which directly follows from the short distance behaviour (A.43) of massive propagators. \square

Usually massless bosons are Wick ordered with respect to mass $M = 1$, which makes the extra power of $1/M$ in (A.46) equal to one. The propagator $C_0(x)$ can be rewritten as $-1/4\pi \ln(\mu^2 x^2)$ which coincides with the expression given in the propagator appendix.

The neutrality condition which was presented as an algebraic identity is a nice consistency check of the formalism for massless scalar fields in two dimensions. The Lagrangian is invariant under

$$\varphi(x) \longrightarrow \varphi(x) + c, \quad (\text{A.48})$$

where c is some constant. The expectation values considered in (A.39) formally transform like

$$\left\langle \prod_{j=1}^n : e^{i\varphi(t_j)} :_M \right\rangle_{C_{m=0}} \longrightarrow \left\langle \prod_{j=1}^n : e^{i\varphi(t_j)} :_M \right\rangle_{C_{m=0}} \exp \left(ic \sum_{j=1}^n q_j \right). \quad (\text{A.49})$$

If now the neutrality condition were not there, the symmetry would be broken, which is not possible since continuous symmetries cannot be broken in two dimensions [19]. Thus one can consider the neutrality condition as a direct consequence of Coleman's theorem.

Appendix B

The technical appendix

The appendix B is a summary of notational conventions and of formulas that can not be found the literature. Since they are of technical nature I did not include them into the main part.

B.1 Notational conventions

γ -algebra :

It is convenient to use the following representation of the 2d Euclidean γ -matrices which makes the fermion propagator antidiagonal

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \gamma_5 = i\gamma_2\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (\text{B.1})$$

They obey the commutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$; $\mu, \nu = 1, 2, 5$. Chiral projectors P_\pm are defined as

$$P_\pm := \frac{1}{2}(1 \pm \gamma_5) . \quad (\text{B.2})$$

Fourier transform:

I use a symmetric normalization of the $1/2\pi$ factors which gives for the Fourier transform in two dimensions

$$\hat{f}(p) := \frac{1}{2\pi} \int d^2x f(x) e^{-ipx} , \quad f(x) = \frac{1}{2\pi} \int d^2p \hat{f}(p) e^{ipx} , \quad (\text{B.3})$$

and formally (correct when smeared with test functions)

$$\delta(x) = \frac{1}{(2\pi)^2} \int d^2p e^{ipx} . \quad (\text{B.4})$$

B.2 Some integrals

The following integrals are used in the main part. They all can be evaluated after a transformation to polar coordinates r, φ . It turned out that integrating over φ first is simpler. For some of the integrals partial integration in r is necessary to bring them into a form such that they can be found in the integral tables [1], [13], [33] and [46]. In all cases it was possible to cross-check the formulas. For properties of special functions I use [43].

I₁ :

$$I_1 := \int d^2p \frac{1}{p^2 + \lambda^2} e^{ipx} = 2\pi K_0(\lambda|x|) , \quad (\text{B.5})$$

where K_0 is the modified Bessel function (see [43] p. 66). Here one should remark, that I_1 is not absolutely convergent, but converges conditionally for $x \neq 0$. Thus some regularization procedure has to be applied. In particular $(p^2 - \lambda^2)^{-1}$ can be replaced by $(p^2 - \lambda^2)^{-\alpha}$ which gives an absolutely convergent integral for $\alpha > 1$ which can be solved explicitly. The result (B.5) is then obtained by analytic continuation to $\alpha = 1$ (see Vol. I of [29] for details).

I₂ :

$$\begin{aligned} I_2 &:= \int d^2p \frac{1}{(p^2 + \lambda^2)p^2} (1 - \cos(px)) \\ &= \frac{2\pi}{\lambda^2} \left(\ln|x| + K_0(\lambda|x|) + \ln\left(\frac{\lambda}{2}\right) + \gamma \right) , \end{aligned} \quad (\text{B.6})$$

where $\gamma = 0.577216..$ denotes Euler's constant.

I₃ :

$$\begin{aligned} I_3 &:= \int d^2p e^{-2\frac{|p|}{n}} \frac{1}{p^2} (1 - \cos(px)) \\ &= 2\pi \ln \left(\frac{n}{4} \left[\frac{2}{n} + \sqrt{\left(\frac{2}{n}\right)^2 + x^2} \right] \right) \\ &= 2\pi \left(\ln|x| + \ln\left(\frac{n}{4}\right) + O\left(\frac{1}{n}\right) \right) \text{ for } n \rightarrow \infty . \end{aligned} \quad (\text{B.7})$$

I₄ :

$$\begin{aligned} I_4 &:= \int d^2p e^{-2\frac{|p|}{n}} \frac{1}{p^2 + \lambda^2} \\ &= -2\pi \left(\cos\left(\frac{2}{n}\lambda\right) \text{Ci}\left(\frac{2}{n}\lambda\right) + \sin\left(\frac{2}{n}\lambda\right) \text{si}\left(\frac{2}{n}\lambda\right) \right) \\ &= 2\pi \left(-\gamma - \ln\left(\frac{2}{n}\lambda\right) + O\left(\frac{1}{n}\right) \right) \text{ for } n \rightarrow \infty . \end{aligned} \quad (\text{B.8})$$

Here $\text{si}(x)$ and $\text{Ci}(x)$ denote the sine and the cosine integral (see [43] p. 347).

I₅ :

$$\begin{aligned} I_5 &:= \int d^2p \frac{1}{(p^2 + \lambda^2)p^2} \sin^2(px) \\ &= \frac{\pi}{\lambda^2} \left(\ln(2|x|) + K_0(\lambda 2|x|) + \ln\left(\frac{\lambda}{2}\right) + \gamma \right). \end{aligned} \quad (\text{B.9})$$

I₆ :

$$\begin{aligned} I_6 &= \int d^2p \frac{1}{p^2 + \lambda^2} \frac{1}{p_2^2} \left(\sin^2(p_2 L) - 2 \sin^2(p_2 L/2) \right) \\ &- 2 \int d^2p \frac{1}{p^2 + \lambda^2} \frac{1}{p_2^2} \cos(p_1 2L) \left(\sin^2(p_2 L) - \sin^2(p_2 L/2) \right). \end{aligned} \quad (\text{B.10})$$

For this integral the large L behaviour is of interest. In both terms the p_2 integral can be solved using the formula

$$\int_0^\infty dx \frac{\sin^2(bx)}{x^2(x^2 + z^2)} = \frac{\pi}{4z^3} (e^{-bz} + 2bz - 1) \quad (\text{B.11})$$

which can be found in [46]. Some terms then cancel each other and the remaining integrals either vanish exponentially with L , or give a constant. One ends up with

$$I_6 = \frac{\pi}{\lambda^2} + O(e^{-L}) \quad \text{for } L \rightarrow \infty. \quad (\text{B.12})$$

B.3 Some matrices

In this appendix I discuss the properties of some matrices acting in flavor space.

$$R := \begin{pmatrix} N-1 & -1 & . & . & -1 \\ -1 & N-1 & -1 & . & . \\ . & -1 & . & . & . \\ . & . & . & . & -1 \\ -1 & . & . & -1 & N-1 \end{pmatrix}. \quad (\text{B.13})$$

R has the following set of orthonormal eigenvectors

$$\vec{r}^{(1)} = c^{(1)} \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ . \\ . \\ 1 \end{pmatrix}, \quad \vec{r}^{(2)} = c^{(2)} \begin{pmatrix} 1 \\ . \\ . \\ . \\ . \\ 1 \\ -(N-1) \end{pmatrix},$$

$$\vec{r}^{(3)} = c^{(3)} \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ -(N-2) \\ 0 \end{pmatrix}, \dots, \vec{r}^{(N)} = c^{(N)} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{pmatrix}, \quad (\text{B.14})$$

where the normalization constants are given by

$$c^{(1)} = \frac{1}{\sqrt{N}}, \quad c^{(2)} = \frac{1}{\sqrt{N-1+(N-1)^2}},$$

$$c^{(3)} = \frac{1}{\sqrt{N-2+(N-2)^2}}, \dots, \quad c^{(N)} = \frac{1}{\sqrt{2}}. \quad (\text{B.15})$$

Obviously the eigenvalues $e^{(I)}$, $I = 1 \dots N$ are given by

$$e^{(1)} = 0, \quad e^{(2)} = N, \quad e^{(3)} = N, \dots, \quad e^{(N)} = N. \quad (\text{B.16})$$

This implies that R can be diagonalized by the matrix U constructed out of the vectors $\vec{r}^{(I)}$, $I = 1 \dots N$

$$U := \left(\vec{r}^{(1)}, \vec{r}^{(2)}, \dots, \vec{r}^{(N)} \right)^T. \quad (\text{B.17})$$

$$URU^T = \text{diag}(0, N, \dots, N). \quad (\text{B.18})$$

U is an orthogonal matrix

$$U^T = U^{-1}. \quad (\text{B.19})$$

Finally I denote the useful identity

$$\sum_{I=2}^N U_{Ia} U_{Ib} = \delta_{ab} - \frac{1}{N}. \quad (\text{B.20})$$

Using the fact that $U_{1a} = 1/\sqrt{N}$ for $a = 1, 2, \dots, N$ and the orthogonality of U one obtains

$$\delta_{ab} = \sum_{I=1}^N U_{Ia} U_{Ib} = \sum_{I=2}^N U_{Ia} U_{Ib} + \frac{1}{N}, \quad (\text{B.21})$$

which implies the quoted identity.

B.4 Inverse conditioning

In this appendix the inverse conditioning formula [26] adapted to the case of several flavors is proven.

Lemma B.1 : Inverse conditioning

Let C^1, C^2 be covariances that obey $C^1 \geq C^2 \geq 0$ as quadratic forms. Then the following inequality holds

$$\begin{aligned} & \left\langle \prod_{b=1}^N \left[\prod_{i_b}^{q_b} : e^{i2\sqrt{\pi}\varphi^{(b)}(x_{i_b}^{(b)})} :_{C^1} \prod_{j_b}^{2n_b-q_b} : e^{-i2\sqrt{\pi}\varphi^{(b)}(y_{j_b}^{(b)})} :_{C^1} \right] \right\rangle_{C^1} \\ & \leq e^{4\pi \sum_{b=1}^N 2n_b \lambda^{(b)}} \left\langle \prod_{b=1}^N \left[\prod_{i_b}^{q_b} : e^{i2\sqrt{\pi}\varphi^{(b)}(x_{i_b}^{(b)})} :_{C^2} \prod_{j_b}^{2n_b-q_b} : e^{-i2\sqrt{\pi}\varphi^{(b)}(y_{j_b}^{(b)})} :_{C^2} \right] \right\rangle_{C^2}, \end{aligned} \quad (\text{B.22})$$

where

$$\lambda^{(b)} := \lim_{z \rightarrow 0} \frac{1}{2} [C_{bb}^1(z) - C_{bb}^2(z)]. \quad (\text{B.23})$$

Proof :

Denote by $\varphi(\delta_n(x))$ the convolution of the field φ with a δ -sequence δ_n peaked at x . This allows to write the left hand side of (B.22) as

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\langle \prod_{b=1}^N \left[\prod_{i_b}^{q_b} : e^{i2\sqrt{\pi}\varphi^{(b)}(\delta_n(x_{i_b}^{(b)}))} :_{C^1} \prod_{j_b}^{2n_b-q_b} : e^{-i2\sqrt{\pi}\varphi^{(b)}(\delta_n(y_{j_b}^{(b)}))} :_{C^1} \right] \right\rangle_{C^1} \\ & = \lim_{n \rightarrow \infty} \left\langle e^{i2\sqrt{\pi}\varphi(f_n)} \right\rangle_{C^1} e^{\frac{1}{2}4\pi \sum_{b=1}^N 2n_b (\delta_n(\xi), C_{bb}^1 \delta_n(\xi))} \\ & = \lim_{n \rightarrow \infty} e^{-\frac{1}{2}4\pi (f_n, C^1 f_n)} e^{\frac{1}{2}4\pi \sum_{b=1}^N 2n_b (\delta_n(\xi), C_{bb}^2 \delta_n(\xi))} e^{\frac{1}{2}4\pi \sum_{b=1}^N 2n_b (\delta_n(\xi), [C_{bb}^1 - C_{bb}^2] \delta_n(\xi))} \\ & \leq \lim_{n \rightarrow \infty} e^{-\frac{1}{2}4\pi (f_n, C^2 f_n)} e^{\frac{1}{2}4\pi \sum_{b=1}^N 2n_b (\delta_n(\xi), C_{bb}^2 \delta_n(\xi))} e^{\frac{1}{2}4\pi \sum_{b=1}^N 2n_b (\delta_n(\xi), [C_{bb}^1 - C_{bb}^2] \delta_n(\xi))} \\ & = e^{4\pi \sum_{b=1}^N 2n_b \lambda^{(b)}} \left\langle \prod_{b=1}^N \left[\prod_{i_b}^{q_b} : e^{i2\sqrt{\pi}\varphi^{(b)}(x_{i_b}^{(b)})} :_{C^2} \prod_{j_b}^{2n_b-q_b} : e^{-i2\sqrt{\pi}\varphi^{(b)}(y_{j_b}^{(b)})} :_{C^2} \right] \right\rangle_{C^2}. \end{aligned} \quad (\text{B.24})$$

f_n denotes the vector composed from the sum over all δ -sequences δ_n peaked at the various space-time arguments. In the last step I used

$$\lambda^{(b)} = \lim_{n \rightarrow \infty} \frac{1}{2} (\delta_n(\xi), [C_{bb}^1 - C_{bb}^2] \delta_n(\xi)), \quad (\text{B.25})$$

which coincides with (B.23). ξ denotes some dummy space-time argument.

□

B.5 Conditioning

In this section the conditioning formula [26], [34] will be proven. Again the result is quoted in the form which is suitable for the N-flavor case.

Lemma B.2 : Conditioning

Let C^1, C^2 be two covariances that obey $C^1 \geq C^2 \geq 0$ as quadratic forms. Then the conditioning formula holds

$$\begin{aligned} & \left\langle \prod_{b=1}^N \exp \left(-2 \sum_{b=1}^N \beta^{(b)} \epsilon^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} \varphi^{(b)}(x) \right] :_{C^2} \right) \right\rangle_{C^2} \\ & \leq \left\langle \prod_{b=1}^N \exp \left(-2 \sum_{b=1}^N \beta^{(b)} \epsilon^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} \varphi^{(b)}(x) \right] :_{C^1} \right) \right\rangle_{C^1}, \quad (\text{B.26}) \end{aligned}$$

where $\epsilon^{(b)} \in \{-1, +1\}$ arbitrary but fixed.

Proof :

The proof makes use of Jensen's inequality (see e.g. [61])

$$\int d\mu[\varphi] \exp(F(\varphi)) \geq \exp \left(\int d\mu[\varphi] F(\varphi) \right), \quad (\text{B.27})$$

whenever $\int d\mu[\varphi] \exp(F(\varphi)) < \infty$. Introducing new fields θ_1 with covariance $C^1 - C^2$ and θ_2 with covariance C^2 , the right hand side of (B.26) can be written as

$$\begin{aligned} & \left\langle \prod_{b=1}^N e^{-2 \sum_{b=1}^N \beta^{(b)} \epsilon^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} (\theta_1^{(b)}(x) + \theta_2^{(b)}(x)) \right] :_{C^1 - C^2, C^2}} \right\rangle_{C^1 - C^2, C^2} \\ & \geq \left\langle \prod_{b=1}^N e^{-2 \int d\mu_{C^1 - C^2}[\theta_1] \sum_{b=1}^N \beta^{(b)} \epsilon^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} (\theta_1^{(b)}(x) + \theta_2^{(b)}(x)) \right] :_{C^1 - C^2, C^2}} \right\rangle_{C^1 - C^2, C^2} \\ & = \left\langle \prod_{b=1}^N e^{-2 \sum_{b=1}^N \beta^{(b)} \epsilon^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} \theta_2^{(b)}(x) \right] :_{C^2}} \right\rangle_{C^2}, \quad (\text{B.28}) \end{aligned}$$

where Jensen's inequality for θ_1 was used in the first step. The second step made use of

$$\begin{aligned} & \int d\mu_{C^1 - C^2}[\theta_1] \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} (\theta_1^{(b)}(x) + \theta_2^{(b)}(x)) \right] :_{C^1 - C^2, C^2} \\ & = \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} \theta_2^{(b)}(x) \right] :_{C^2}, \quad (\text{B.29}) \end{aligned}$$

which can be seen to hold from the definition of Wick ordering (A.34).

□

From Lemma B.2 one easily reads off the following corollary by expressing the cosh in terms of exponentials and inserting the corresponding values for $\epsilon^{(b)} \in \{-1, +1\}$.

Corollary B.1 :

For $C_1 \geq C_2$ the following inequality holds

$$\begin{aligned} & \left\langle \prod_{b=1}^N 2 \cosh \left(-2 \sum_{b=1}^N \beta^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} \varphi^{(b)}(x) \right] :_{C^2} \right) \right\rangle_{C^2} \\ & \leq \left\langle \prod_{b=1}^N 2 \cosh \left(-2 \sum_{b=1}^N \beta^{(b)} \int_{\Lambda} d^2 x \, t(x) : \cos \left[2\sqrt{\pi} \varphi^{(b)}(x) \right] :_{C^1} \right) \right\rangle_{C^1} . \quad (\text{B.30}) \end{aligned}$$

B.6 Dirichlet boundary conditions

In this appendix formulas for covariances with Dirichlet boundary conditions are collected. They all are discussed in the proof for one flavor by Fröhlich [25], [26].

By scaling one may choose for the space-time cutoff Λ a unit square, and by Euclidean invariance of the measure one may suppose that Λ is centered at $(1,0)$ with sides parallel to the coordinate axes. Let S be the disc of radius 2 centered at $(0,0)$. The geometry is illustrated in Figure B.1.

Figure B.1 : The boundary ∂S and the space-time region Λ .

As can be evaluated easily

$$\text{dist}(\Lambda, \partial S) = 2 - \frac{\sqrt{10}}{2} > 0. \quad (\text{B.31})$$

Let Δ_S be the Laplacian on $L^2(S, d^2x)$ with zero Dirichlet data on ∂S . Because of zero Dirichlet data on the boundary, $-\Delta_S$ is strictly positive, and hence gives rise to a proper covariance $(-\Delta_S)^{-1}$. Furthermore

$$\frac{1}{-\Delta_S + M^2}, \quad (\text{B.32})$$

is a proper covariance operator as well. Obviously as an operator on $L^2(S, d^2x)$

$$\frac{1}{-\Delta_S} \geq \frac{1}{-\Delta_S + M^2}, \quad (\text{B.33})$$

for some real mass M , and finally

$$\frac{1}{-\Delta + M^2} \geq \frac{1}{-\Delta_S + M^2}, \quad (\text{B.34})$$

which follows from the fact that Δ is not strictly positive. Furthermore since $\text{dist}(\Lambda, \partial S) > 0$

$$\sup_{x, y \in \Lambda} \left[\frac{1}{-\Delta + M^2}(x, y) - \frac{1}{-\Delta_S + M^2}(x, y) \right] \leq \tilde{\omega} < \infty. \quad (\text{B.35})$$

Using the method of image charges one can construct an explicit representation of the Green's function $(-\Delta_S)^{-1}(x, y) =: C^{0,S}(x, y)$

$$(-\Delta_S)^{-1}(x, y) = -\frac{1}{4\pi} \left(\ln |\tilde{x} - \tilde{y}| + \ln |\hat{x} - \hat{y}| - \ln |\hat{x} - \tilde{y}| - \ln |\tilde{x} - \hat{y}| \right), \quad (\text{B.36})$$

where

$$\tilde{x} := x_1 + ix_2, \quad (\text{B.37})$$

denotes the complex coordinate already encountered in (A.14), and

$$\hat{x} := 4 / \bar{\tilde{x}}, \quad (\text{B.38})$$

is the reflection of \tilde{x} at the circle ∂S .

Finally

$$|\tilde{x} - \hat{x}| < 8, \quad (\text{B.39})$$

for $x \in \Lambda$. This can easily be seen to hold from the definition of \tilde{x} , \hat{x} and the relative position of Λ and ∂S (compare Figure B.1).

B.7 A generalized Hölder inequality

In this section a generalization of Hölder's inequality is proven (This generalization is an exercise in [23]). Furthermore I infer a corollary that is needed in the main text.

Lemma B.3 : (generalized Hölder inequality)

Let for positive numbers $0 < q_1, q_2, \dots, q_n < \infty$

$$\frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_n} = 1, \quad (\text{B.40})$$

and

$$f_i(x) \in L^{q_i}(\mathbb{R}^D), \quad i = 1, 2, \dots, n, \quad (\text{B.41})$$

where the number of dimensions D is a positive but arbitrary integer. Then

$$\prod_{i=1}^n f_i(x) \in L^1(\mathbb{R}^D), \quad (\text{B.42})$$

and

$$\int d^D x \prod_{i=1}^n |f_i(x)| \leq \prod_{i=1}^n \|f_i\|_{q_i}, \quad (\text{B.43})$$

where

$$\|f\|_q := \left(\int d^D x |f(x)|^q \right)^{\frac{1}{q}}. \quad (\text{B.44})$$

Proof: (by induction)

i: For $n = 2$ the claim reduces to the usual Hölder inequality.

ii: Let the lemma be true for $n - 1$.

iii: n :

$q_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ obeying (B.40) and $f_i \in L^{q_i}(\mathbb{R}^D)$, $i = 1, 2, \dots, n$ are given. Define

$$\frac{1}{p} := \frac{1}{q_n}, \quad \frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_{n-1}}. \quad (\text{B.45})$$

Obviously

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{q_1/q} + \dots + \frac{1}{q_{n-1}/q} = 1. \quad (\text{B.46})$$

From $f_i(x) \in L^{q_i}(\mathbb{R}^D)$, $i = 1, \dots, n-1$ there follows $|f_i(x)|^q \in L^{q_i/q}(\mathbb{R}^D)$, $i = 1, \dots, n-1$. Thus the assumption for $n-1$ and the usual Hölder inequality can be applied to finish the proof,

$$\int d^D x \prod_{i=1}^n |f_i(x)| \leq \left[\int d^D x \prod_{i=1}^{n-1} |f_i(x)|^q \right]^{\frac{1}{q}} \left[\int d^D x |f_n(x)|^p \right]^{\frac{1}{p}} \leq$$

$$\prod_{i=1}^{n-1} \left[\int d^D x |f_i(x)|^{qq_1/q} \right]^{\frac{1}{qq_1/q}} \left[\int d^D x |f_n(x)|^p \right]^{\frac{1}{p}} = \prod_{i=1}^n \|f_i\|_{q_i} . \quad (\text{B.47})$$

□

Lemma B.3 allows to prove a corollary that will be used in the main part.

Corollary B.2 :

Denote by $f^{(a,b)}$, $1 \leq a < b \leq N$ functions depending on the coordinates

$$z_1^{(a)} , \dots , z_{n_a}^{(a)} , \quad z_1^{(b)} , \dots , z_{n_b}^{(b)} , \quad (\text{B.48})$$

where each $z_j^{(c)}$ is itself D -dimensional. Assume

$$f^{(a,b)} \in L^{N-1} \left(\mathbb{R}^{D(n_a+n_b)} , \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{j=1}^{n_b} d^D z_j^{(b)} \right) , \quad (\text{B.49})$$

for $1 \leq a < b \leq N$. Then

$$\prod_{a < b}^N f^{(a,b)} \in L^1 \left(\mathbb{R}^{D \sum_{a=1}^N n_a} , \prod_{b=1}^N \prod_{j=1}^{n_b} d^D z_j^{(b)} \right) , \quad (\text{B.50})$$

and

$$\int \prod_{a=1}^N \prod_{j=1}^{n_a} d^D z_j^{(a)} \left| \prod_{b < c}^N f^{(b,c)} \right| \leq \prod_{a < b}^N \left[\int \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{j=1}^{n_b} d^D z_j^{(b)} \left| f^{(a,b)} \right|^{N-1} \right]^{\frac{1}{N-1}} . \quad (\text{B.51})$$

Proof :

The statement will be proven by direct construction. For convenience I introduce the notations

$$\|f^{(a,b)}\|_b := \left[\int \prod_{j=1}^{n_b} d^D z_j^{(b)} \left| f^{(a,b)} \right|^{N-1} \right]^{\frac{1}{N-1}} , \quad (\text{B.52})$$

and

$$\|f^{(a,b)}\| := \left[\int \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{j=1}^{n_b} d^D z_j^{(b)} \left| f^{(a,b)} \right|^{N-1} \right]^{\frac{1}{N-1}} . \quad (\text{B.53})$$

The latter is of course the usual Hölder norm $\|f\|_{N-1}$. From (B.49) and Definition (B.52) there follows immediately that

$$\|f^{(a,b)}\|_b \in L^{N-1} \left(\mathbb{R}^{D n_a} , \prod_{i=1}^{n_a} d^D z_i^{(a)} \right) . \quad (\text{B.54})$$

Then

$$\begin{aligned}
& \int \prod_{a=1}^N \prod_{i=1}^{n_a} d^D z_i^{(a)} \left| \prod_{b < c}^N f^{(b,c)} \right| \\
&= \int \prod_{a=1}^{N-1} \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{b < c}^{N-1} \left| f^{(b,c)} \right| \int \prod_{j=1}^{n_N} d^D z_j^{(N)} \prod_{d=1}^{N-1} \left| f^{(d,N)} \right| \\
&\leq \int \prod_{a=1}^{N-1} \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{b < c}^{N-1} \left| f^{(b,c)} \right| \prod_{d=1}^{N-1} \left\| f^{(d,N)} \right\|_N \\
&= \int \prod_{a=1}^{N-2} \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{b < c}^{N-2} \left| f^{(b,c)} \right| \int \prod_{j=1}^{n_{N-1}} d^D z_j^{(N-1)} \prod_{d=1}^{N-2} \left| f^{(d,N-1)} \right| \prod_{e=1}^{N-1} \left\| f^{(e,N)} \right\|_N \\
&\leq \int \prod_{a=1}^{N-2} \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{b < c}^{N-2} \left| f^{(b,c)} \right| \prod_{d=1}^{N-2} \left\| f^{(d,N-1)} \right\|_{N-1} \left\| f^{(d,N)} \right\|_N \times \left\| f^{(N-1,N)} \right\| \\
&\leq \int \prod_{a=1}^{N-3} \prod_{i=1}^{n_a} d^D z_i^{(a)} \prod_{b < c}^{N-3} \left| f^{(b,c)} \right| \prod_{d=1}^{N-3} \left\| f^{(d,N-2)} \right\|_{N-2} \left\| f^{(d,N-1)} \right\|_{N-1} \left\| f^{(d,N)} \right\|_N \\
&\quad \times \left\| f^{(N-2,N-1)} \right\| \left\| f^{(N-2,N)} \right\| \left\| f^{(N-1,N)} \right\| \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\leq \int \prod_{a=1}^2 \prod_{i=1}^{n_a} d^D z_i^{(a)} \left| f^{(1,2)} \right| \prod_{b=1}^2 \prod_{c=3}^N \left\| f^{(b,c)} \right\|_c \times \prod_{d=3}^N \prod_{e=d+1}^N \left\| f^{(d,e)} \right\| \\
&\leq \int \prod_{i=1}^{n_1} d^D z_i^{(1)} \prod_{a=2}^N \left\| f^{(1,a)} \right\|_a \times \prod_{b=2}^N \prod_{c=b+1}^N \left\| f^{(b,c)} \right\| \\
&= \prod_{a < b}^N \left\| f^{(a,b)} \right\|. \tag{B.55}
\end{aligned}$$

In the chain of inequalities above, I successively applied the generalized Hölder inequality (B.43) with

$$\frac{1}{q_i} = \frac{1}{N-1}, \quad i = 1, \dots, N-1. \tag{B.56}$$

□

B.8 Bound on integrals over Cauchy determinants

The following bound on integrals over Cauchy determinants can be found in [26]. I quote it for the convenience of the reader.

Lemma B.4 :

For $\alpha < 1$ the following bound holds

$$\int_{\Lambda} \left[\prod_{i=1}^n d^2 x_i d^2 y_i \right] \left| \det_{i,j=1,\dots,2n} \left(\frac{1}{w_i - z_j} \right) \right|^\alpha \leq (2n)! [\Xi(\alpha)]^{2n}, \quad (\text{B.57})$$

where

$$w := \begin{pmatrix} \tilde{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{x}_n \\ \hat{y}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{y}_n \end{pmatrix}, \quad z := \begin{pmatrix} \tilde{y}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{y}_n \\ \hat{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_n \end{pmatrix}. \quad (\text{B.58})$$

For the definitions of \tilde{x} and \hat{x} see (B.37) and (B.38).

Proof :

$$\begin{aligned} & \int_{\Lambda} \left[\prod_{i=1}^n d^2 x_i d^2 y_i \right] \left| \det_{i,j=1,\dots,2n} \left(\frac{1}{w_i - z_j} \right) \right|^\alpha \\ &= \int_{\Lambda} \left[\prod_{i=1}^n d^2 x_i d^2 y_i \right] \left| \sum_{\pi(2n)} \text{sign}(\pi) \prod_{j=1}^{2n} \frac{1}{w_j - z_{\pi(j)}} \right|^\alpha \\ &\leq \sum_{\pi(2n)} \int_{\Lambda} \left[\prod_{i=1}^n d^2 x_i d^2 y_i \right] \prod_{j=1}^{2n} \left| \frac{1}{w_j - z_{\pi(j)}} \right|^\alpha \\ &\leq \sum_{\pi(2n)} [\Xi(\alpha)]^{2n} = (2n)! [\Xi(\alpha)]^{2n}. \end{aligned} \quad (\text{B.59})$$

In the last step I used that $\prod_{j=1}^{2n} \left| \frac{1}{w_j - z_{\pi(j)}} \right|^\alpha$ is $\prod_{i=1}^n d^2 x_i d^2 y_i$ integrable for $\alpha < 1$ and

$$\int_{\Lambda} \left[\prod_{i=1}^n d^2 x_i d^2 y_i \right] \prod_{j=1}^{2n} \left| \frac{1}{w_j - z_{\pi(j)}} \right|^\alpha \leq [\Xi(\alpha)]^{2n}, \quad (\text{B.60})$$

for some constant $\Xi(\alpha)$, independent of the permutation π (see [26] for details).

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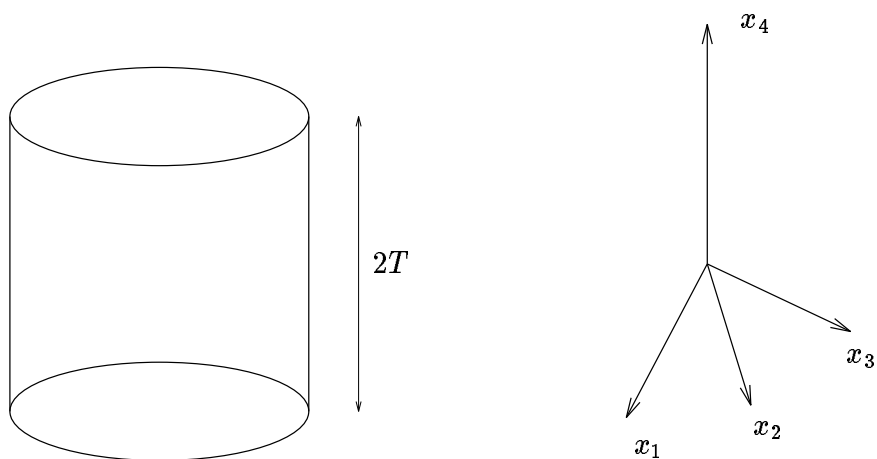


Figure 2.1 : The surface ∂V_4 .

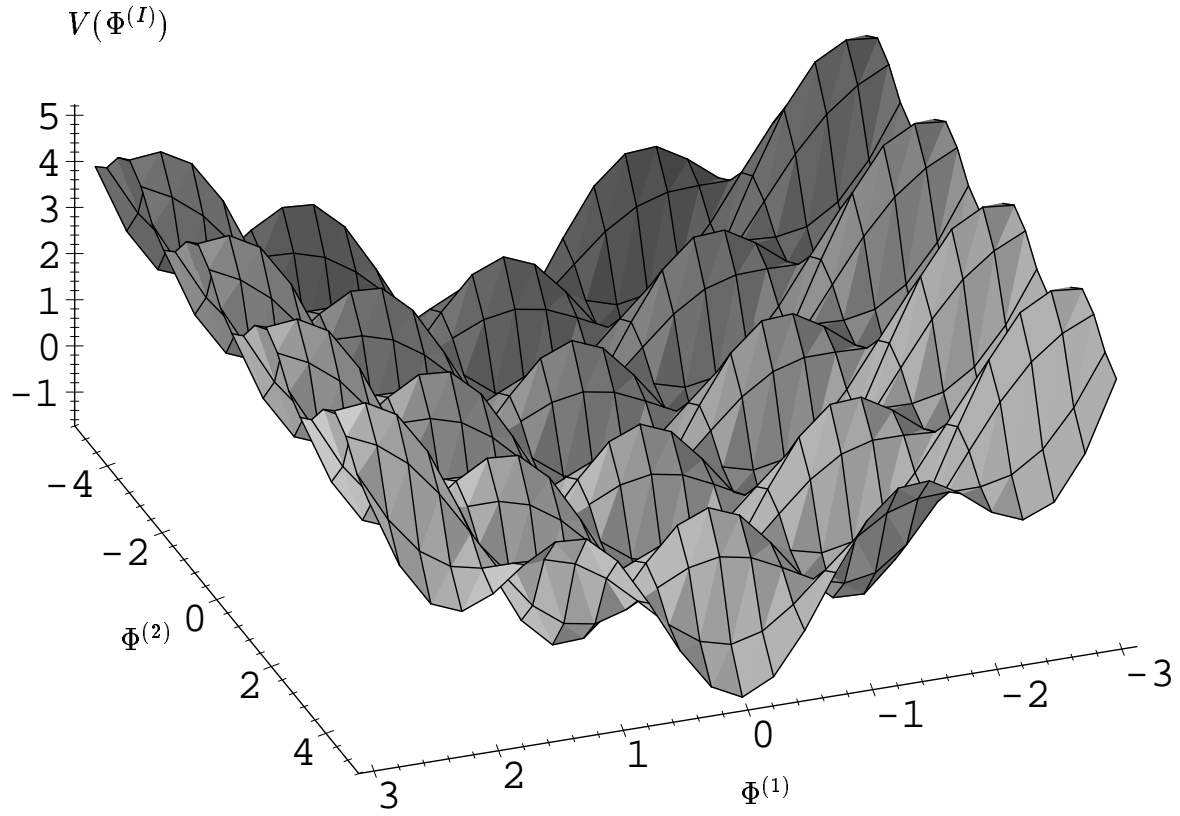


Figure 7.1 : Plot of the potential $V(\Phi^{(I)})$ defined in Equation (7.123) for $N = 2$ flavors. The values of the parameters are given in the text.

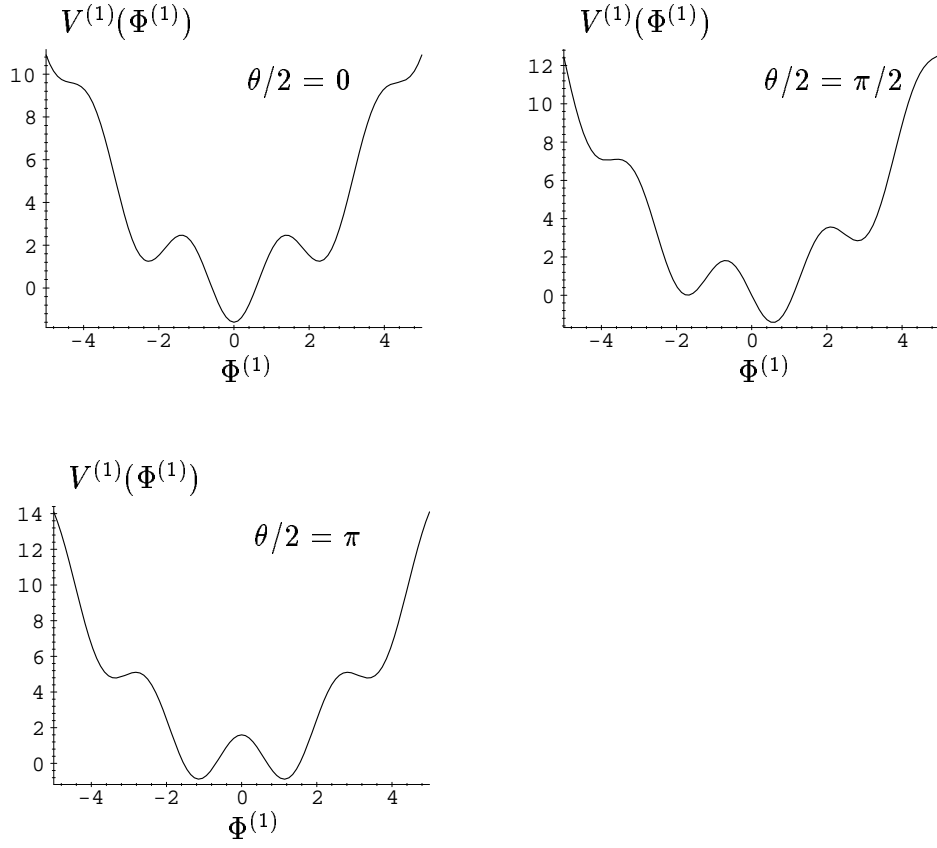


Figure 7.2 : Plot of the Potential $V^{(1)}(\Phi^{(1)})$ defined in (7.137) for $N = 2$ flavors and $\theta/2 = 0$, $\theta/2 = \pi/2$ and $\theta/2 = \pi$. The values of the other parameters are given in the text.

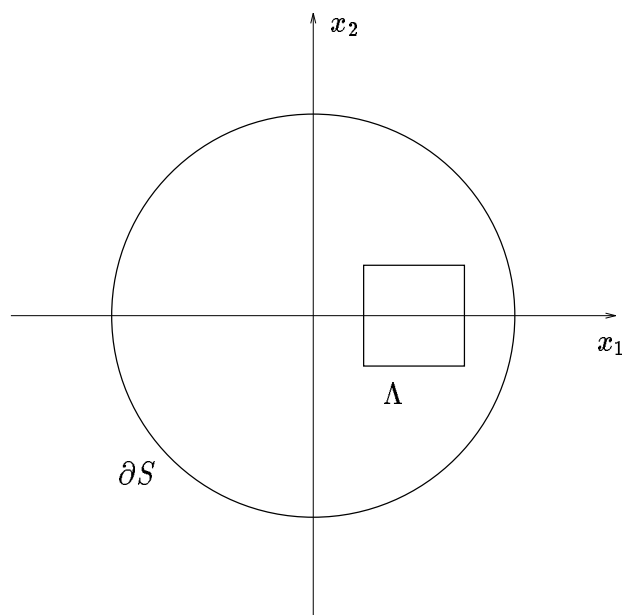


Figure B.1 : The boundary ∂S and the space-time region Λ .

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